Timelike Circular Orbits and Efficiency of Compact Objects

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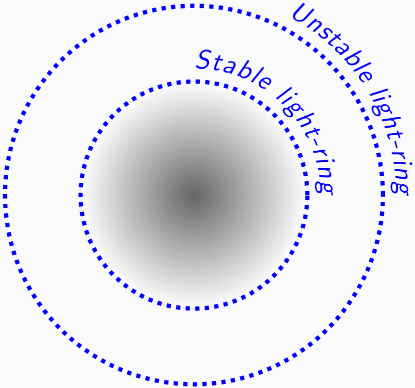
YouTube link to video of the talk:
`https://youtu.be/tZCnJnj-aCo`

Motivation

Ultra-compact Horizonless objects

Phys.Rev.Lett. 119 (2017) 25, 251102



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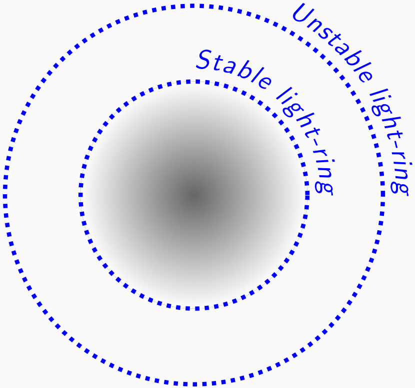
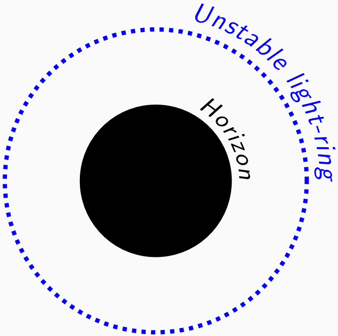


Black Holes

Phys.Rev.Lett. 124 (2020) 18, 181101



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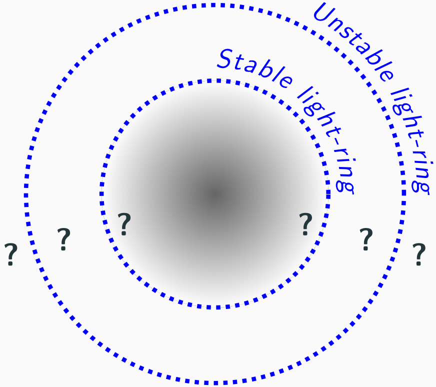

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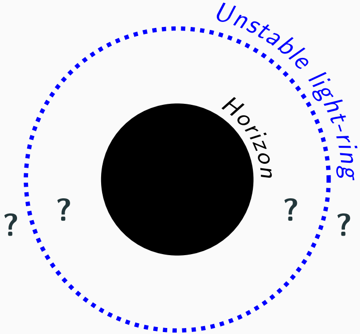

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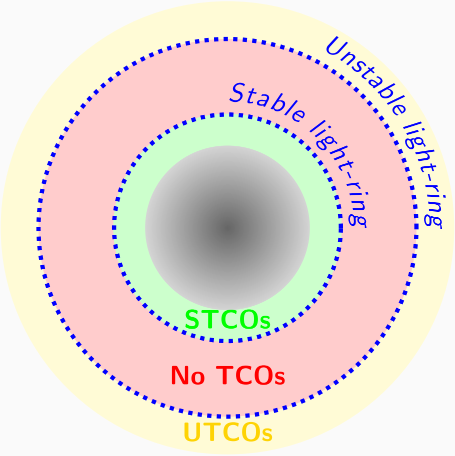

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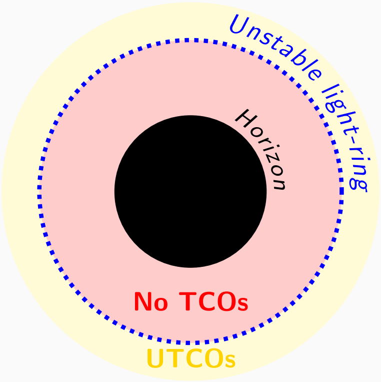

Generic Spacetime

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Due to the symmetries, together with some gauge choices, we can write,

$$\boxed{ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\varphi}(r,\theta)dtd\varphi + g_{\varphi\varphi}(r,\theta)d\varphi^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2}$$

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We also assume a $\mathbb{Z}_2$ symmetry $\longrightarrow$ $\theta = \pi/2$ plane is a totally geodesic submanifold.

Circular Causal Orbits on the Equatorial Plane $\theta = \pi/2$

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Effective Lagrangian of a test particle,

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \xi \ , \quad \xi \equiv \begin{cases} -1 \ , & \text{timelike} \\ 0 \ , & \text{null} \\ 1 \ , & \text{spacelike} \end{cases}$$

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Effective potential $V_\xi(r)$,

$$\boxed{V_\xi(r) \equiv g_{rr}\dot{r}^2 = \xi + \frac{A(r,E,L)}{B(r)}}$$

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A particle will follow a circular orbit at $r = r^{\text{cir}}$ iff,

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The radial stability of such orbit can be verified by the sign of $V''_\xi(r^{\text{cir}})$,

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$V''_\xi(r^{\text{cir}}) > 0$

Unstable Circular Orbits

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For a generic stationary spacetime we can find pairs of solutions corresponding to *co-rotating* orbits $(r^{\text{cir}}_+, E_+, L_+)$ and *counter-rotating* orbits $(r^{\text{cir}}_-, E_-, L_-)$.

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Solving both equations gives the *inverse impact parameter* $\sigma_\pm = E_\pm/L_\pm$ and the radial coordinate of the light-ring, $r = r^{\mathrm{LR}}$.

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The radial stability of the light-ring is evaluated by checking the sign of $V_0''(r^{\mathrm{LR}})$,

$$V_0''(r^{\mathrm{LR}}) = L_\pm^2 \left[\frac{g''_{\varphi\varphi}\sigma_\pm^2 + 2g''_{t\varphi}\sigma_\pm + g''_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}\right]_{\mathrm{LR}}$$

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Positive Numerator

Unstable Light Ring

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Positive Numerator	Negative Numerator
Unstable Light Ring	Stable Light Ring

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$$C(r) = (g'_{t\varphi})^2 - g'_{tt}g'_{\varphi\varphi}$$

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Radial Stability, $$V''_{-1}(r^{\text{cir}}) = \left[\frac{g''_{\varphi\varphi}E_\pm^2 + 2g''_{t\varphi}E_\pm L_\pm + g''_{tt}L_\pm^2 - (g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})''}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}\right]_{r^{\text{cir}}}$$

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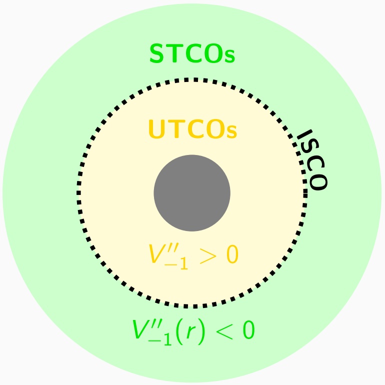

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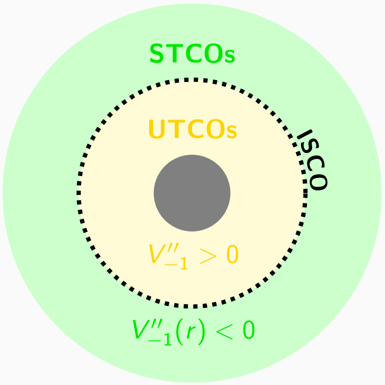

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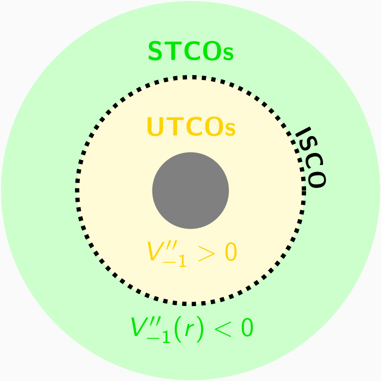

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Standard ISCO

Solution with the largest r such that,

$$V''_{-1}(r^{\text{ISCO}_{\text{std}}}) = 0 \quad \wedge \quad V'''_{-1}(r^{\text{ISCO}_{\text{std}}}) < 0$$

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If $C(r^{\text{cir}}) < 0$ then no circular (timelike, null or spacelike) geodesics are possible since $\Omega_{\pm} \in \mathbb{C}$

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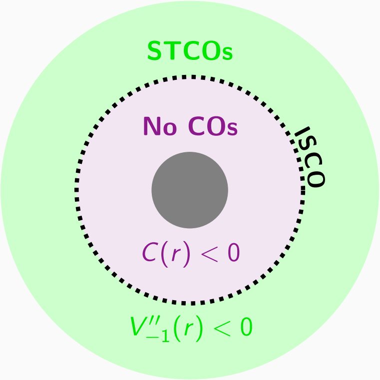

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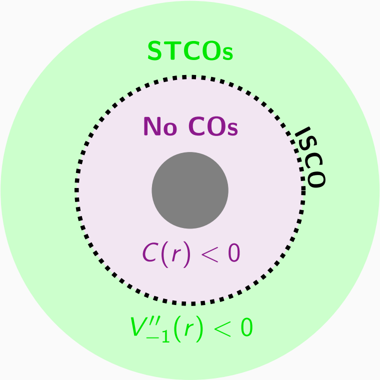

Absolute ISCO

Solution with the smallest r such that,

$$C(r^{\text{ISCO}_{\text{abs}}}) = 0 \quad \wedge \quad V''_{-1}(r^{\text{ISCO}_{\text{abs}}} + |\delta r|) < 0\ , \ \ \delta r \ll 1$$

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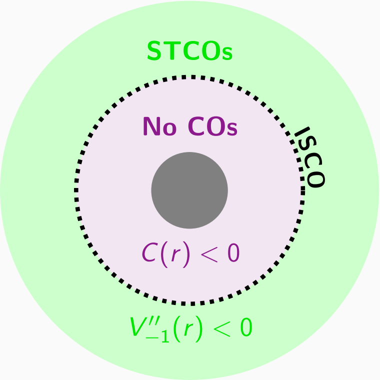

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or the solution with the smallest r such that,

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Main Result

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Thus,

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$$\boxed{\beta_\pm(r) = \frac{V_0''(r_\pm^{\text{LR}})}{L_\pm^2}\left[\frac{\left(g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}\right)^3}{(g'_{t\varphi})^2 - g'_{tt}g'_{\varphi\varphi}}\right]_{\text{LR}}^{1/2} \delta r + \mathcal{O}(\delta r^2)}$$

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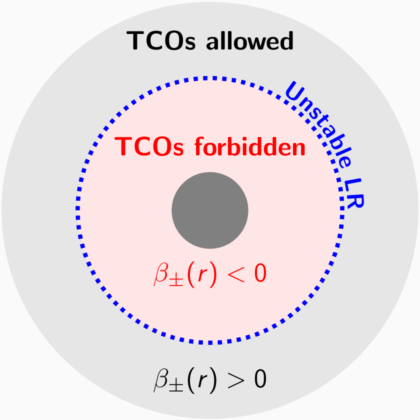
Unstable light-ring: $V_0''(r_\pm^{\mathrm{LR}}) > 0$



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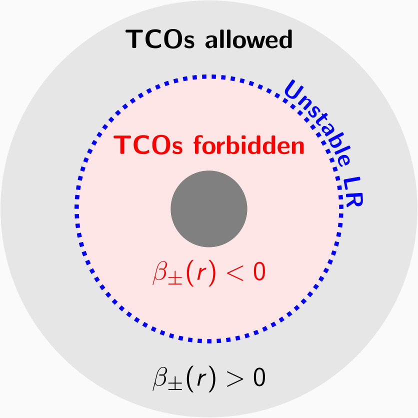

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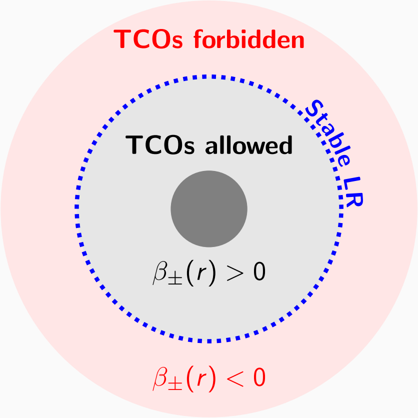

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Thus, the sign of the numerator is dictated by the stability of the light-ring, $V_0''(r_\pm^{\text{LR}})$.

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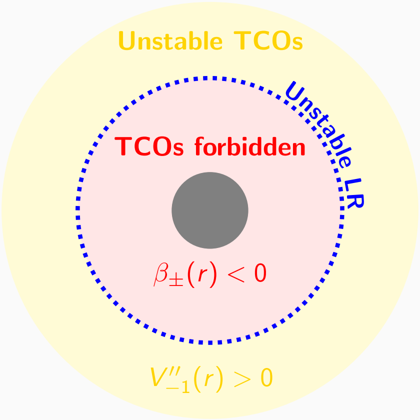

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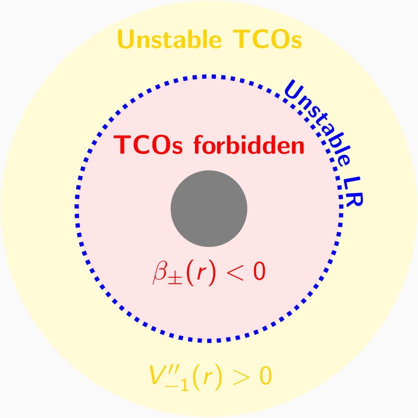

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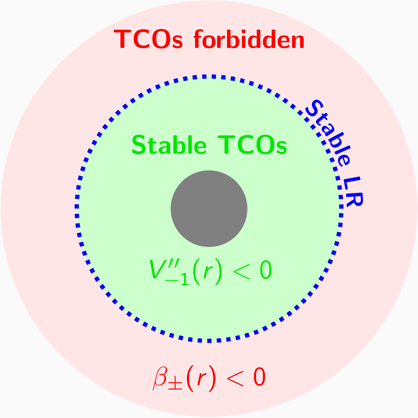
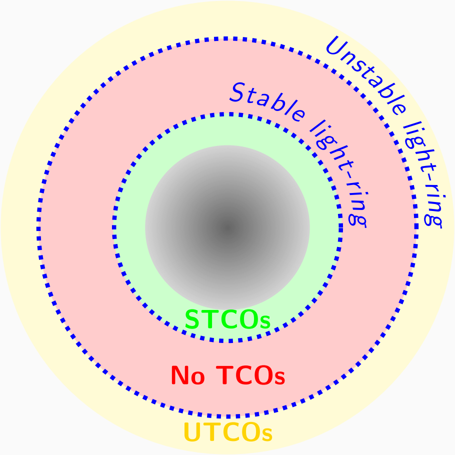


Corollaries

Ultra-compact Horizonless objects



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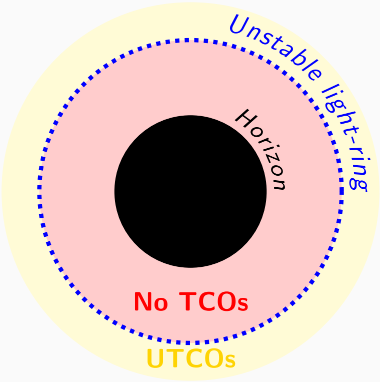

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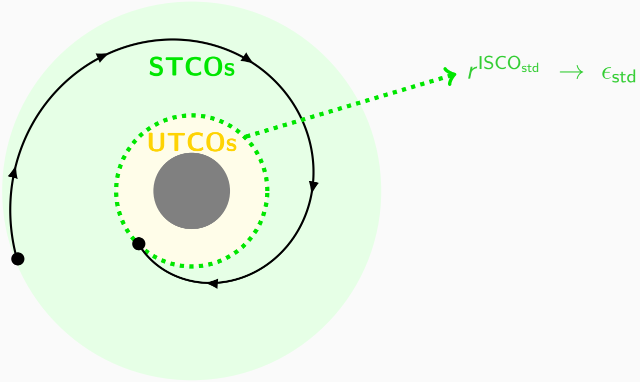

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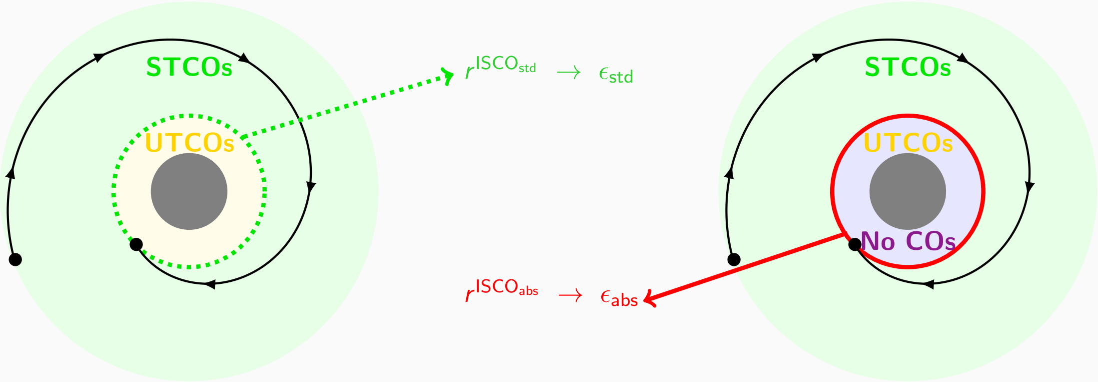

Stars

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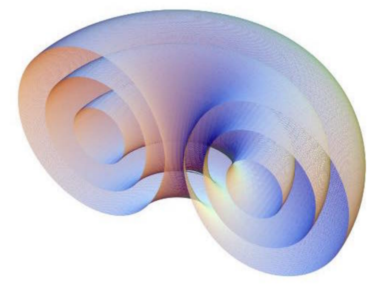
Spinning Scalar Boson Stars

$$\Psi = \phi e^{i(m\varphi - \omega t)}$$



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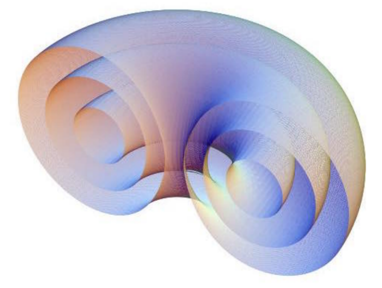

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Spinning Vector Boson Stars

$$A = \left(iVdt + \frac{H_1}{r} dr + H_2 d\theta + iH_3 \sin\theta d\varphi \right) e^{i(m\varphi - \omega t)}$$

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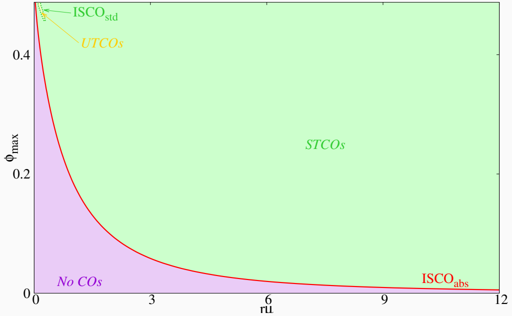
Co-rotating orbits

Structure of Circular Orbits



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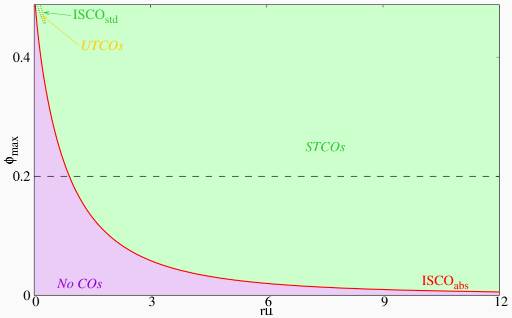

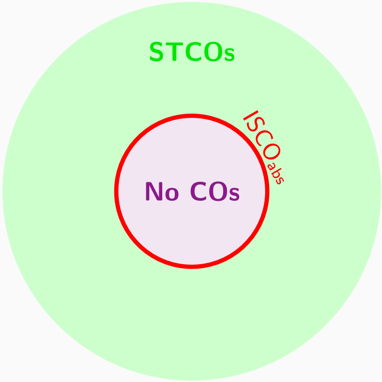

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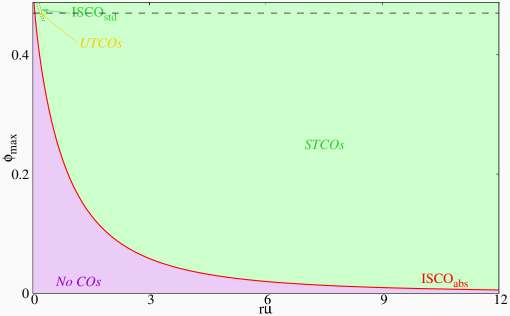

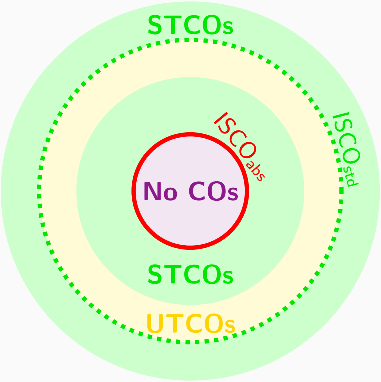

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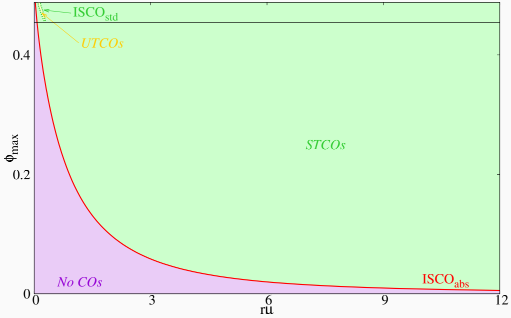

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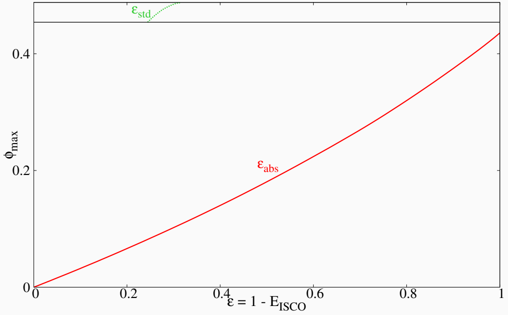

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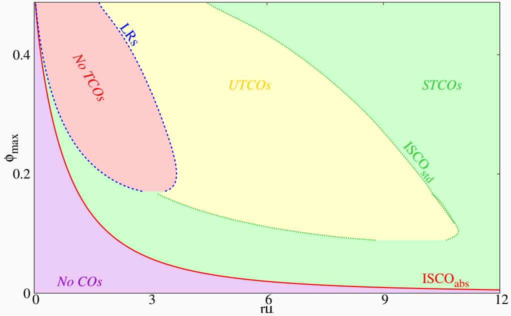

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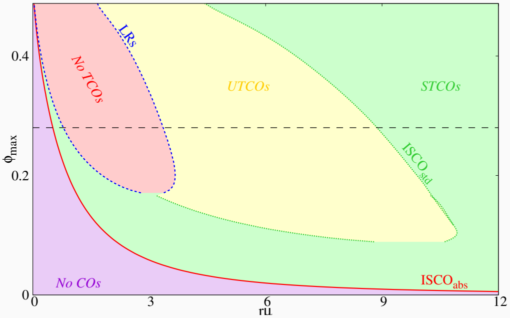

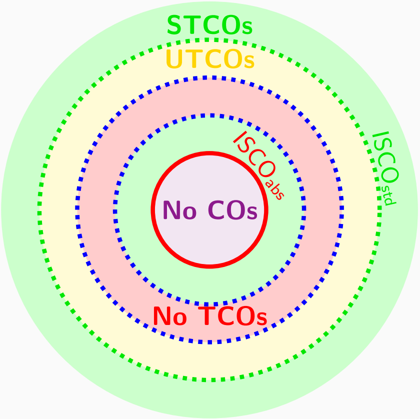

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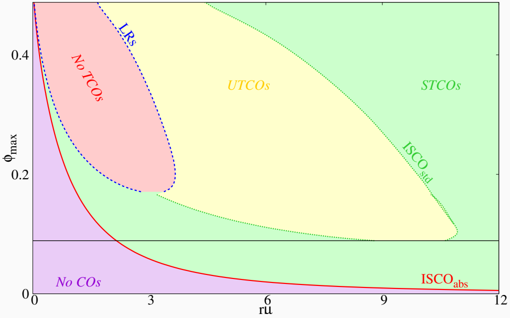

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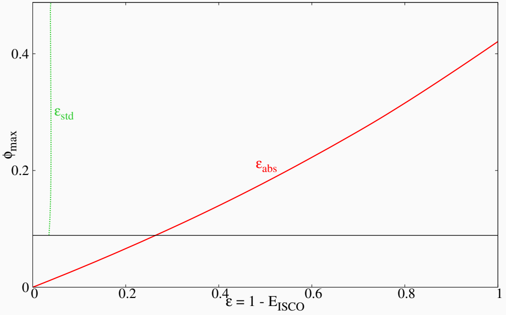

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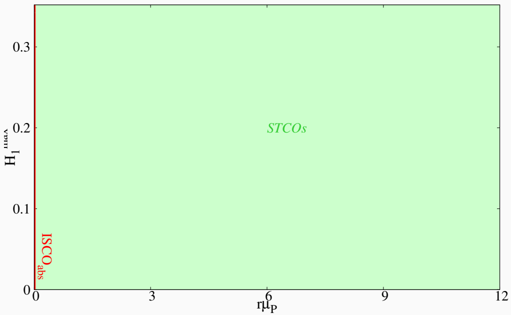

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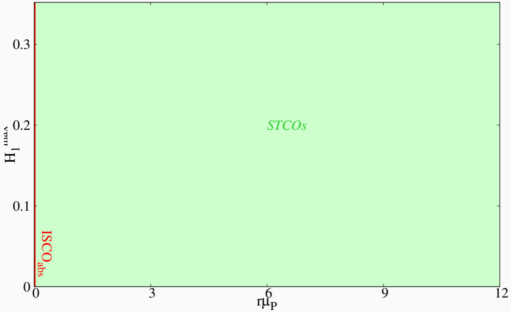

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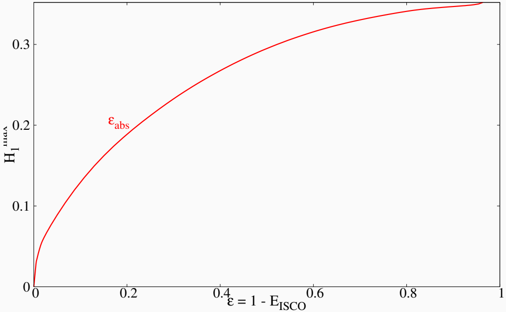

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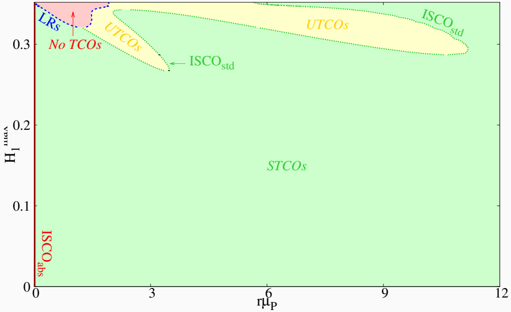

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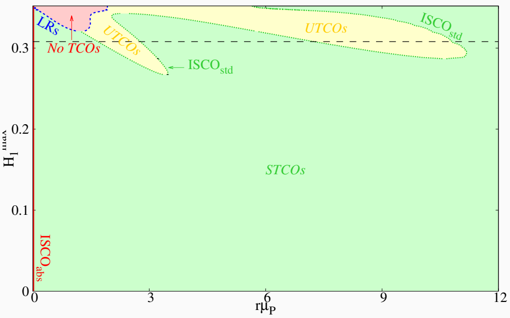

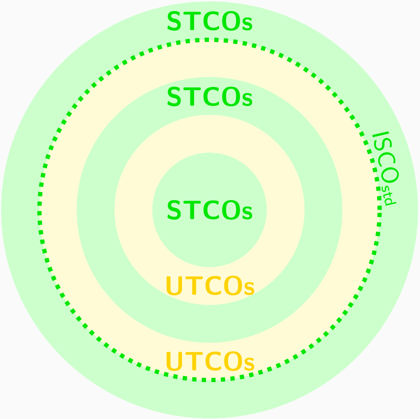

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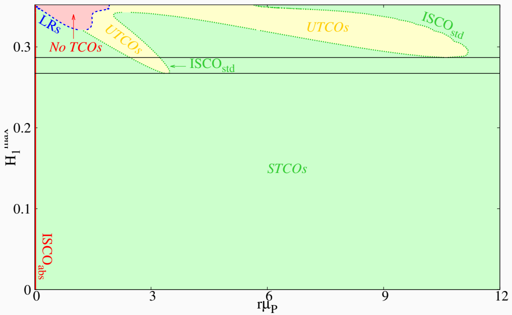

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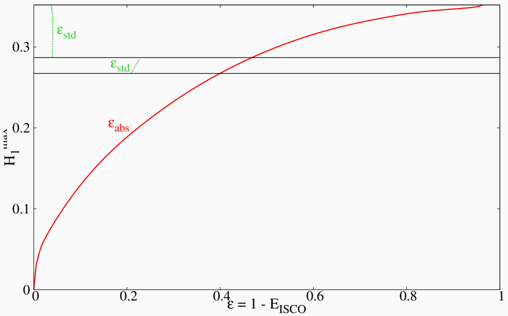

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Final Remarks - Timelike Circular Orbits

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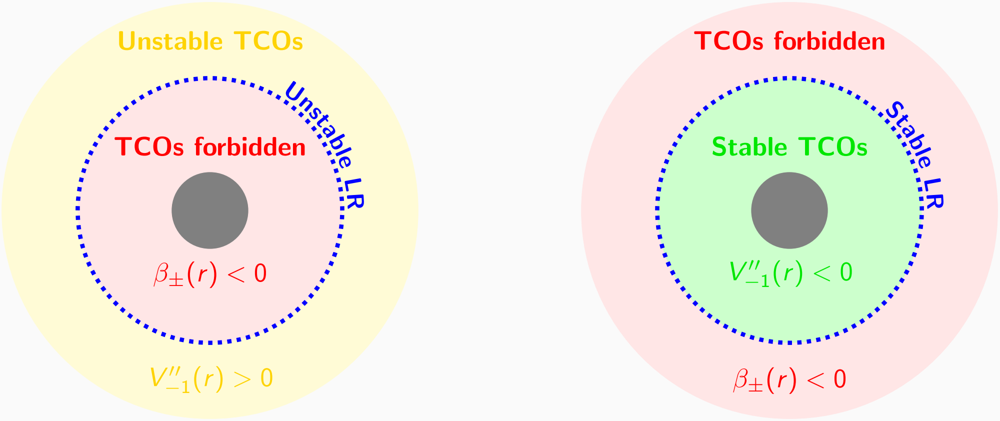
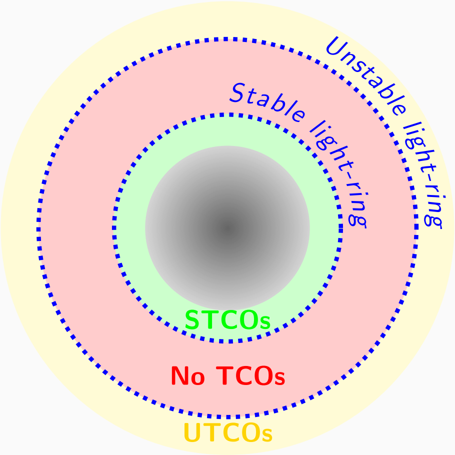
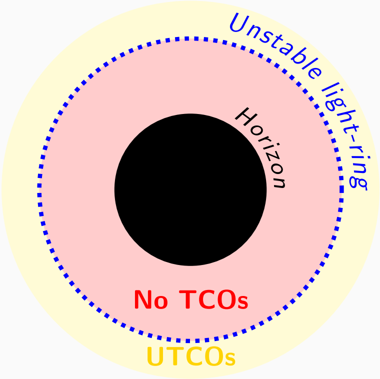


Final Remarks - Timelike Circular Orbits

Ultra-compact Horizonless objects



Black Holes



Final Remarks - Efficiency

Spinning Scalar Boson Stars

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Spinning Scalar Boson Stars

Absolute Efficiency ϵ_{abs}

$\rightarrow$ For both co- and counter-rotating orbits, the efficiency can grow arbitrarily close to unity.

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Standard Efficiency ϵ_{std}

$\rightarrow$ Co-rotating orbits show larger efficiencies than counter-rotating orbits.

Final Remarks - Efficiency

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Spinning Vector Boson Stars

Absolute Efficiency ϵ_{abs}

→ For both co- and counter-rotating orbits, it is possible to have stable TCOs until the center of the star.

→ The efficiency increases monotonically towards values close to 100%.

Final Remarks - Efficiency

Spinning Scalar Boson Stars

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Standard Efficiency ϵ_{std}

→ For counter-rotating orbits, more compact stars can have disconnected regions of unstable TCOs.

→ The efficiency can drop to small values.

Thank you.

jorgedelgado@ua.pt

universidade de aveiro

MARIE CURIE ACTIONS

FCT
Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

Generic Spacetime

$(\mathcal{M}, g)$ is a stationary, axi-symmetric, asymptotically flat and 1+3 dimensional spacetime.

- Two Killing vectors: $\{\eta_1, \eta_2\} \xrightarrow[\text{flatness}]{\text{asymptotically}} [\eta_1, \eta_2] = 0.$
- Appropriated coordinate system (t, r, θ, φ) such that $\eta_1 = \partial_t$ and $\eta_2 = \partial_\varphi$.

We assume,

1. A north-south $\mathbb{Z}_2$ symmetry.
2. Circularity. $\longrightarrow \quad g_{\rho t} = g_{\rho\varphi} = 0 \ , \ \rho = \{r, \theta\}$

Gauge choice:

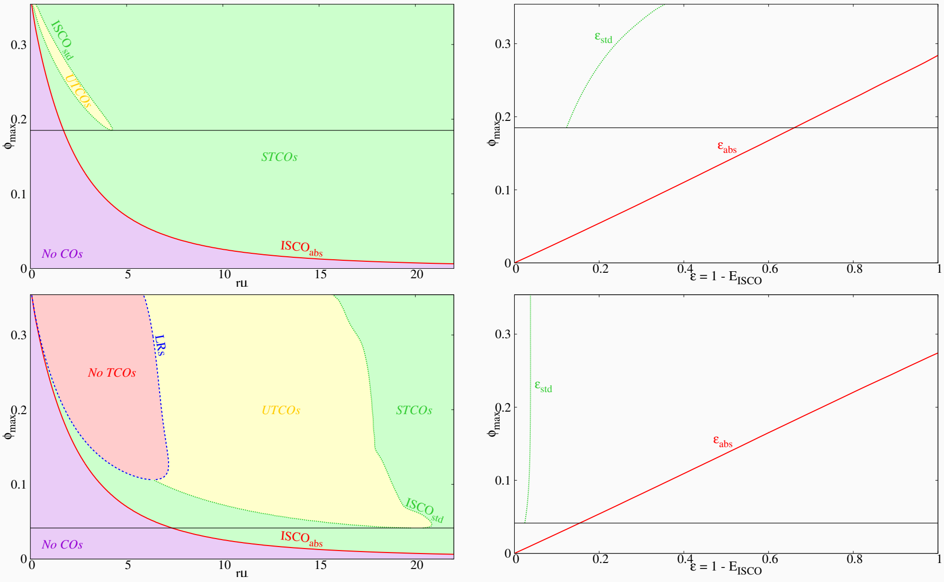
- ⋆ r and θ are orthogonal.
- ⋆ Horizon located at constant radial coordinate: $r = r_H$.

$\longrightarrow \quad g_{r\theta} = 0 \ , \ g_{rr} > 0 \ , \ g_{\theta\theta} > 0$

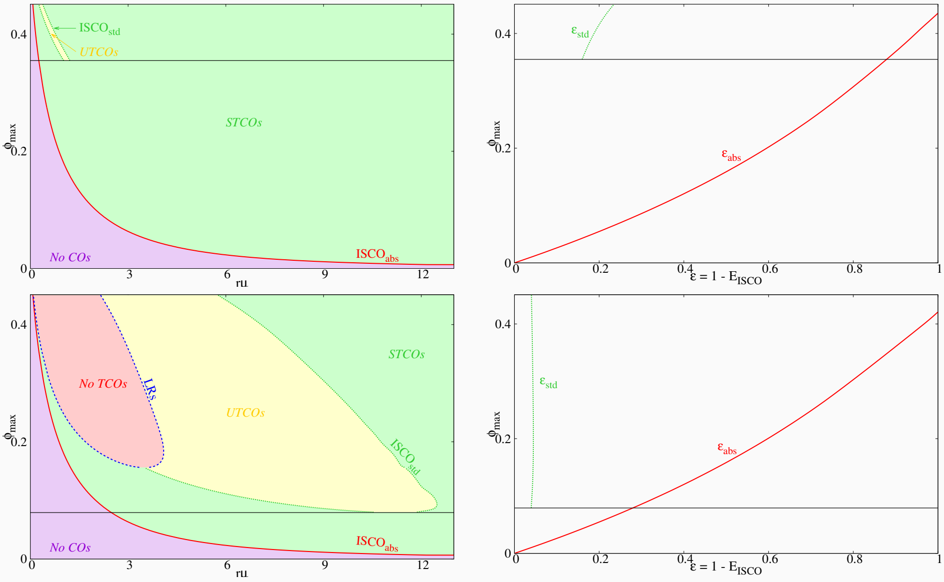
Causality implies $g_{\varphi\varphi} \geq 0$

$$\boxed{ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\varphi}(r,\theta)dtd\varphi + g_{\varphi\varphi}(r,\theta)d\varphi^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2}$$

Mini-Boson Stars $m = 2$



Gauged Boson Stars $q_E = 0.6$



Axion Boson Stars $f_a = 0.05$

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi} - g^{\mu\nu}\partial_\mu\Psi^*\partial_\nu\Psi - V(|\Psi^2|)\right] \ , \qquad V(\phi) = \frac{2\mu_a^2 f_a}{B}\left[1 - \sqrt{1 - 4B\sin^2\left(\frac{\phi}{2f_a}\right)}\right]$$

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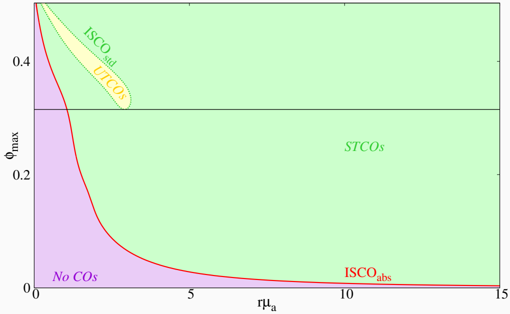
Co-rotating orbits

Structure of Circular Orbits



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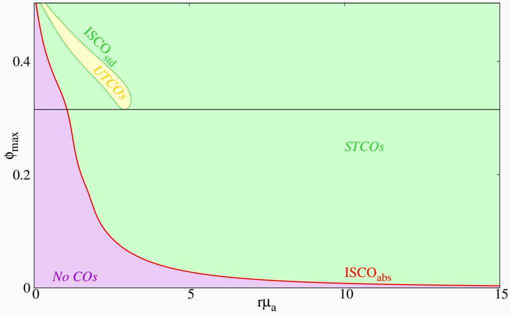
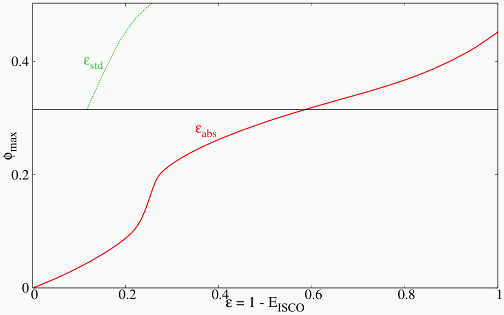


Efficiency



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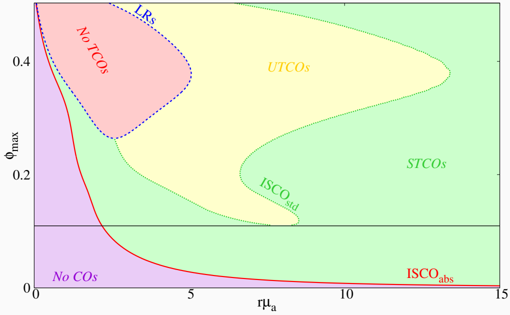
Counter-rotating orbits

Structure of Circular Orbits



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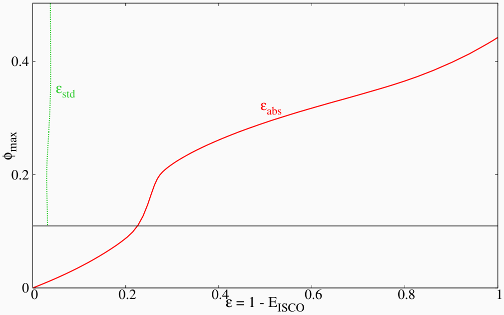
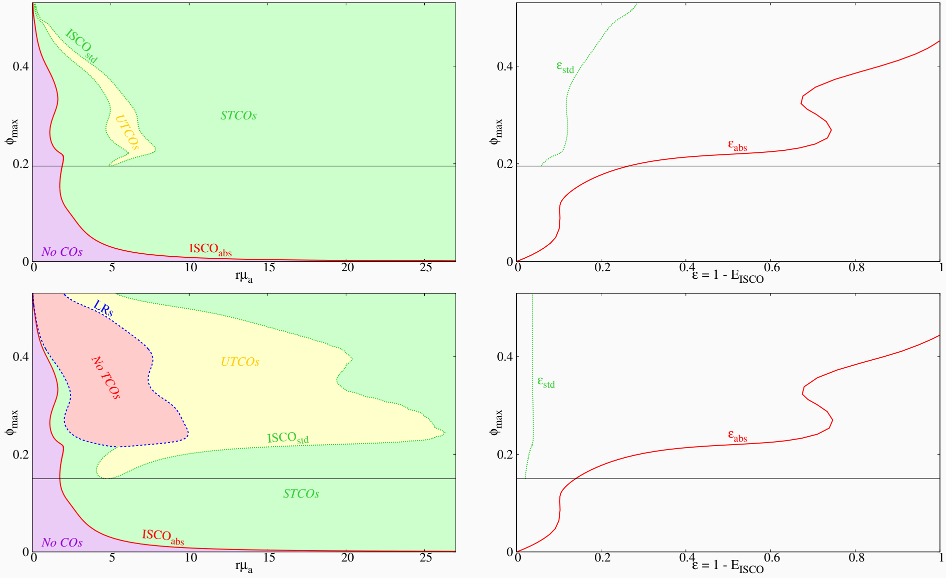


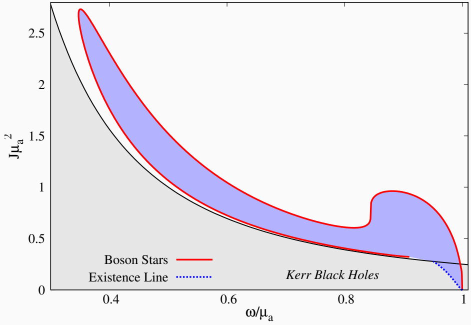
Axion Boson Stars $f_a = 0.03$



Black Holes

Kerr Black Holes with Synchronised Axionic Hair

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi} - g^{\mu\nu}\partial_\mu\Psi^*\partial_\nu\Psi - V(|\Psi^2|)\right]$$



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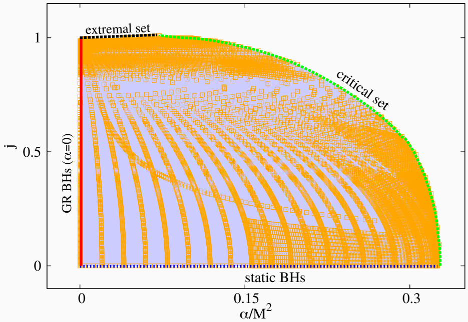
Einstein-scalar-Gauss-Bonnet Black Holes

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\phi\partial^\nu\phi + \alpha\phi R^2_{GB}\right]$$



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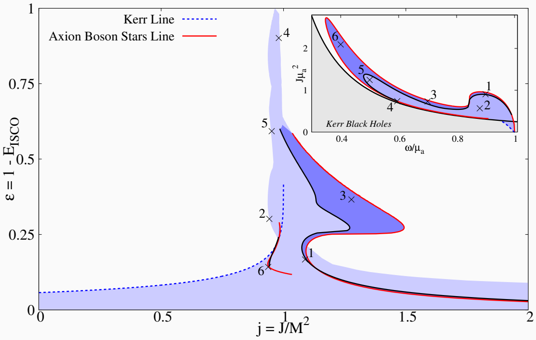
Co-rotating orbits



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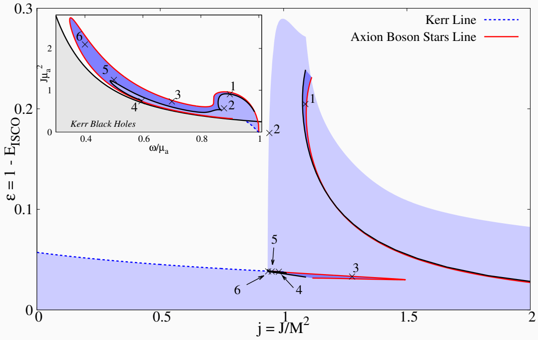

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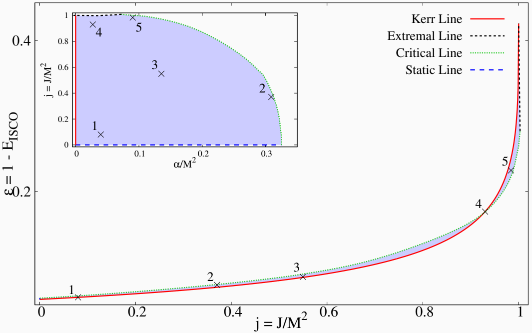

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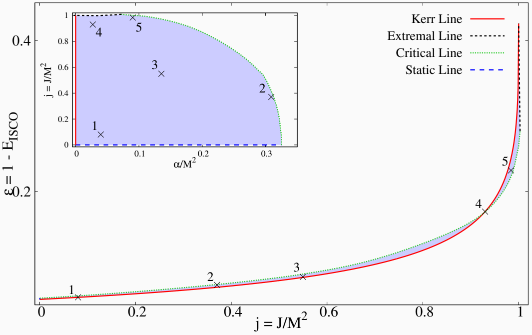

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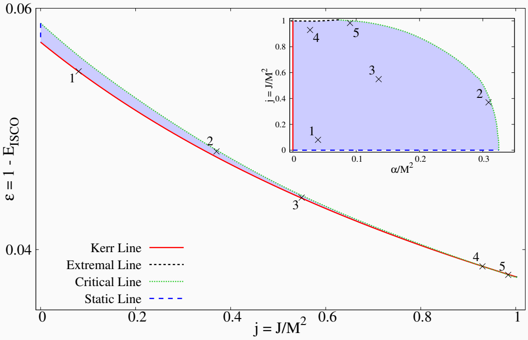

Final Remarks - Efficiency

Kerr Black Holes with Synchronised Axionic Hair

New disconnected regions of unstable and no TCOs develop.

Co-rotating orbits (Ω_+)

→ The efficiency can be much larger than the maximal efficiency for Kerr black holes and can grow close to the unity.

Counter-rotating orbits (Ω_-)

→ The efficiency is smaller than the one for co-rotating orbits, but it can be higher than the maximal efficiency for (counter-rotating) Kerr black holes.

The structure of circular orbits is identical to Kerr black holes.

Co-rotating orbits (Ω_+)

→ For small j, the efficiency is slightly larger than a Kerr black hole with the same j

→ The opposite happens for large j.

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→ The efficiency is larger than their Kerr counterpart, but the differences decreases as j increases.

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