Eikonal quasinormal modes and shadow of string-corrected d-dimensional black holes

Filipe Moura

Departament of Mathematics, ISCTE - University Institute of Lisbon,
Instituto de Telecomunicações, Lisbon, Portugal.
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Perturbations on the $(d-2)$-sphere

- For a system with **spherical symmetry** (metric and all other fields);
- Metric of the type
  $ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\,d\Omega^2_{d-2}$;
- General tensors of rank at least 2 on the $(d-2)$-sphere can be uniquely decomposed in their *tensorial, vectorial and scalar* components;
- One can in general consider perturbations to the metric and any other physical field of the system under consideration;
- Variation of the metric

$$h_{\mu\nu} = \delta g_{\mu\nu}.$$

The Master Equation

The perturbation/field equation can be written as a "master equation"

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial t^2} =: V\Phi.$$

- $d\,x/d\,r = 1/f$ ("tortoise" coordinate);
- Φ : "master" variable;
- V : potential;
- both Φ and V depend on the type of gravitational perturbations or field considered.

QNM solutions of the master equation

- We consider solutions of the form $\Phi(x,t) = e^{i\omega t}\phi(x)$;
- Master equation is then written in the Schrödinger form

$$\left[-\frac{d^2}{dx^2} + V\right]\phi(x) = \omega^2\phi(x).$$

- **Quasinormal modes**: solutions with **complex frequencies** subject to the boundary conditions

$$\phi \propto e^{-i\omega x}\,,\ x \to +\infty;\ \phi \propto e^{i\omega x}\,,\ x \to -\infty;$$

- They are given in terms of a **multipole number** ℓ and an **overtone number** n.

The eikonal limit $\ell \to +\infty$

For d-dimensional asymptotically flat spherically symmetric BHs in Einstein gravity (Cardoso et. al. 2009):

- $V_0^{\text{eik}}[f(r)] = \ell^2 \frac{f(r)}{r^2}$ (for **all** perturbations);
- eikonal QNMs: $\omega_n = \ell\Omega - i\left(n + \frac{1}{2}\right)\Lambda;$
- $\Omega = \frac{\sqrt{f(r_c)}}{r_c}$: angular velocity at circular null geodesics;
- $\Lambda = \sqrt{-\frac{r_c^2}{2f(r_c)} \frac{d^2}{dx^2}\left(\frac{f}{r^2}\right)_{r=r_c}}$: principal Lyapunov exponent corresponding to such orbit.
- r_c: radius of the circular null geodesic, verifying $2f(r_c) = r_c f'(r_c)$, maximizing $V_0^{\text{eik}}[f(r)]$.

Leading α' corrections

- Effective action in the Einstein frame

$$\frac{1}{2\kappa^2}\int\sqrt{-g}\left[\mathcal{R}-\frac{4}{d-2}\left(\partial^\mu\phi\right)\partial_\mu\phi+\mathrm{e}^{\frac{4}{2-d}\phi}\frac{\lambda}{2}\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}\right]\mathrm{d}^dx,$$

$$\lambda=\frac{\alpha'}{2},\frac{\alpha'}{4}\ \text{(bosonic, heterotic).}$$

- Field equations

$$\mathcal{R}_{\mu\nu} \quad + \quad \lambda\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\mu\rho\sigma\tau}\mathcal{R}_\nu^{\ \rho\sigma\tau}-\frac{1}{2(d-2)}g_{\mu\nu}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0;$$

$$\nabla^2\phi \quad - \quad \frac{\lambda}{4}\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0.$$

The string-corrected tensor potential

$$
\begin{aligned}
V_{\mathsf{T}}[f(r)] \quad &= \quad f(r)\left(\frac{\ell\left(\ell+d-3\right)}{r^2}+\frac{\left(d-2\right)\left(d-4\right)f(r)}{4r^2}+\frac{2\left(d-3\right)\left(1-f(r)\right)}{r^2}+\frac{\left(d-6\right)f'(r)}{2r}\right)\\
&+ \quad \frac{\lambda}{r^2}\left[\left(\frac{2\ell\left(\ell+d-3\right)}{r}+\frac{\left(d-4\right)\left(d-5\right)f(r)}{r}+\frac{\left(d-3\right)\left(1-f(r)\right)}{r}+\left(d-4\right)f'(r)\right)\right.\\
&\times \quad \left(2\frac{1-f(r)}{r}+f'(r)\right)+\left(3(d-3)-(4d-13)f(r)\right)\frac{f'(r)}{r}-\\
&- \quad \left. 4\left(f'(r)\right)^2+\left(d-4\right)f(r)f''(r)-\frac{\left(rf''(r)\right)^2}{d-2}\right]f(r).
\end{aligned}
$$

- This potential for tensor–type gravitational perturbations was obtained by perturbing the $\lambda-$corrected metric field equation (Moura 2013).
- In classical EH gravity ($\lambda=0$) it is the same as the potential for **test** scalar fields (Cardoso/Lemos 2002).

The Callan-Myers-Perry black hole

- Solution of the d-dimensional effective action (CMP 1989) with

$$f(r) = f_0(r)\left(1 + \frac{\lambda}{R_H^2}\delta f(r)\right),\; f_0(r) = 1 - \frac{R_H^{d-3}}{r^{d-3}},$$

$$\delta f(r) = -\frac{(d-3)(d-4)}{2}\,\frac{R_H^{d-3}}{r^{d-3}}\,\frac{1 - \frac{R_H^{d-1}}{r^{d-1}}}{1 - \frac{R_H^{d-3}}{r^{d-3}}}.$$

- $\alpha' = 0$: Schwarzschild-Tangherlini solution;
- black hole temperature:

$$T = \frac{d-3}{4\pi R_H}\left(1 - \frac{(d-1)\,(d-4)}{2}\,\frac{\lambda}{R_H^2}\right).$$

Eikonal limit (tensorial perturbations)

- $V_{\mathsf{T}}^{\mathsf{eik}}[f(r)] = \ell^2 \frac{g(r)}{r^2}$,
  $g(r) \equiv f(r)\left(1 + \frac{2\lambda}{r^2}\left(2\left(1 - f(r)\right) + r f'(r)\right)\right)$;
- maximization of $V_{\mathsf{T}}^{\mathsf{eik}}$ no longer defines a circular null geodesic: $2g(r_t) = r_t g'(r_t)$;
- eikonal QNMs: $\omega_n^{\mathsf{T}} = \ell \Omega_{\mathsf{T}} - i\left(n + \frac{1}{2}\right)\Lambda_{\mathsf{T}}$;
- $\Omega_{\mathsf{T}} = \frac{\sqrt{g(r_t)}}{r_t}$, $\Lambda_{\mathsf{T}} = \sqrt{-\frac{r_t^2}{2g(r_t)} \frac{d^2}{dx^2}\left(\frac{g}{r^2}\right)_{r=r_t}}$.
- Concretely,

$$\Omega_{\mathsf{T}} = 2\pi T \sqrt{\frac{2}{d-3}} \left(\frac{2}{d-1}\right)^{\frac{d-1}{2(d-3)}} \left[1 + 8\pi^2 T^2 \lambda \frac{d-2}{(d-3)^2}\left(d - 4 + 3\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right],$$

$$\Lambda_{\mathsf{T}} = 4\pi T \left(\frac{2}{d-1}\right)^{\frac{d-1}{2(d-3)}} \left[1 + 8\pi^2 T^2 \lambda \frac{(d-2)(d-4)}{(d-3)^2}\left(1 - \left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right].$$

Eikonal limit (scalar test fields)

- Same formulas as Einstein gravity (with implicit α' corrections);
- eikonal QNMs: $\omega_n = \ell\Omega - i\left(n+\frac{1}{2}\right)\Lambda$. Concretely,

$$\Omega = 2\pi T\sqrt{\frac{2}{d-3}}\left(\frac{2}{d-1}\right)^{\frac{d-1}{2(d-3)}}\left[1+8\pi^2T^2\lambda\frac{d-4}{(d-3)^2}\left(d-2+\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right];$$

$$\Lambda = \frac{4\pi T}{\sqrt{2}}\left(\frac{2}{d-1}\right)^{\frac{d-1}{2(d-3)}}\left[1+8\pi^2T^2\lambda\frac{(d-2)(d-4)}{(d-3)^2}\left(1-\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right].$$

- $\Omega \neq \Omega_{\mathsf{T}}$ (isospectrality is raised), but
- $\Lambda = \Lambda_{\mathsf{T}} + \mathcal{O}(\lambda^2)$.
- Is this equality valid for other perturbations?
- Is this equality valid to higher orders in α'?

Shadow cast by the black hole

- For asymptotically flat spherically symmetric BHs (including higher derivative corrections):
- the silhouette forms a disk whose radius is related to the real part of the eikonal QNM frequencies of test scalar fields as (Singh/Ghosh 2018)

$$R_S = \frac{1}{\Omega}.$$

- For the CMP black hole,

$$R_S = \sqrt{\frac{2}{d-3}} \left(\frac{d-1}{2}\right)^{\frac{d-1}{2(d-3)}} \left[1 + \frac{\lambda}{R_H^2} \frac{d-4}{2} \left(1 - \left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right] R_H \, .$$

Some speculations

- Physical quantities such as the black hole mass and temperature can be expressed in terms of R_S and be estimated, if the shadow radius of a black hole turns to be measured.
- Eikonal QNM frequencies can, therefore, be estimated knowing R_S. Since they can also be independently measured, these estimates can be confronted with experimental results, at least in some limits.

Conclusions

- We computed QNM frequencies in the eikonal limit corresponding to test fields and tensorial gravitational perturbations for a $d-$dimensional spherically symmetric black hole with α' corrections in d dimensions.
- Differently than in Einstein gravity, the real parts of these frequencies are no longer equal.
- The corresponding imaginary parts remain equal to the principal Lyapunov exponent corresponding to circular null geodesics, to first order in α'.
- We computed the radius of the shadow cast by these black holes.
- Further work: the asymptotic limit (see João Rodrigues' talk).