Null hypersurfaces and conformal vector fields

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Cyriaque Atindogbé

Department of Mathematics. University of Abomey-Calavi, Benin.

Benjamín Olea

Dpto. Matemática Aplicada. Universidad de Málaga.

Definition (Null hypersurface)

Let (M, g) be a Lorentzian manifold. A hypersurface L is a null hypersurface if g restricted to $T_x L$ is degenerated for each $x \in L$.

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Definition (Rigging vector field)

A vector field $\zeta : L \to TM$ defined over L is a rigging for L if

$$\zeta_x \notin T_xL$$

for all $x \in L$.

A fixed rigging ζ for L induces:

- A **rigged vector field** $\xi \in \mathcal{X}(L)$. It is the unique such that:
  - $g(\xi, \xi) = 0$.
  - $g(\xi, \zeta) = 1$.

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- A **null transverse** vector field given by $N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi$, which holds
  - $g(N, N) = 0$.
  - $g(N, \zeta) = g(N, \xi) = 1$.
  - $N_x \perp \mathcal{S}_x$.
  - $T_x M = T_x L \oplus span(N_x)$.

Fundamental tensors

Given $U, V \in \mathfrak{X}(L)$ we define

$$\begin{aligned} B(U,V) &= -g(\nabla_U \xi, V), \\ C(U,V) &= -g(\nabla_U N, P(V)), \\ \tau(U) &= g(\nabla_U \zeta, \xi), \end{aligned}$$

where $P : TL \to \mathcal{S}$ is the canonical projection.

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Definition (Null mean curvature)

The null mean curvature at $x \in L$ is

$$H_x = \sum_{i=1}^{n-2} B(e_i, e_i),$$

where $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis of $\mathcal{S}_x$.

Induced connections ∇^L and ∇^*

Given $U, V \in \mathfrak{X}(L)$ and $X \in \Gamma(\mathcal{S})$

$$\nabla_U V = \underbrace{\nabla^L_U V}_{\in TL} + B(U,V)N \qquad \nabla_U N = \tau(U)N - \underbrace{A(U)}_{\in TL}$$

$$\nabla_U \xi = -\tau(U)\xi - \underbrace{A^*(U)}_{\in \mathcal{S}} \qquad \nabla^L_U X = \underbrace{\nabla^*_U X}_{\in \mathcal{S}} + C(U,X)\xi$$

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- $B(U,V) = g(A^*(U), V)$.
- $C(U,V) = g(A(U), V)$.

Rigged metric

The metric $\widetilde{g}$ given by

$$\widetilde{g}(U, V) = g(U, V) + g(\zeta, U)g(\zeta, V)$$

for all $U, V \in \mathfrak{X}(L)$ is a Riemannian metric on L called rigged metric.

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Theorem

Let (M,g) be a geodesically null complete Lorentzian manifold with $n > 3$ such that

- There exists a timelike conformal vector field.
- It holds the null convergence condition, $Ric(u,u) \geq 0$ for all null vector u.

Then any totally umbilic null hypersurface with never vanishing null mean curvature which is strongly inextensible is contained in a null cone.

Definition

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Definition

K is called closed and conformal if $\varphi = 0$. In this case $K^\perp$ is integrable.

Examples

- Let $(M, g) = \big(I \times F, \varepsilon dt^2 + f(t)^2 g_F\big)$, with $\varepsilon = \pm 1$. The vector field $K = f\partial t$ is closed, conformal and $g(K, K) = \varepsilon$.

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- Let $(M, g) = \big(\mathbb{R}^2, E(v)du^2 + 2dudv\big)$ and

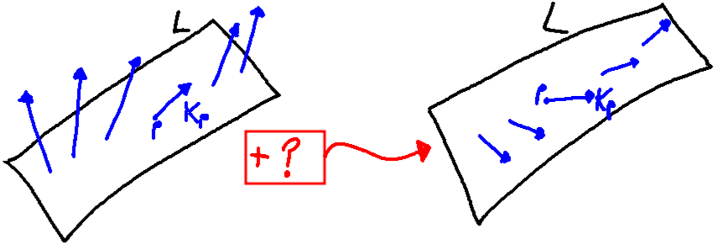
$$K = \partial u - E(v)\partial v.$$

We have $\nabla_U K = -\frac{E_v}{2} U$ for all $U \in \mathfrak{X}(M)$ and thus K is closed and conformal. Moreover $g(K, K) = -E(v)$.

Question

Given a conformal vector field K which is tangent to a null hypersurface L at some point $p \in L$,

under what conditions K is tangent to L at every point?



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- $d\tau = 0$.
- $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.
- $0 \leq trace(A^* \circ A)$.
- $K(span(\xi, N)) \leq Ric(\xi, N)$.

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If $g(K, \xi)$ is signed and there is a point $p \in L$ with $K_p \in T_pL$, then $K_x \in T_xL$ for all $x \in L$.

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Sketch of the proof.

Compute $\widetilde{\triangle} g(K, \xi)$ and apply the maximum principle.

Corollary

Let $K \in \mathfrak{X}(M)$ be a Killing vector field. Suppose that L is a null hypersurface with zero null mean curvature and ζ is a rigging for L such that

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If K_x is causal for all $x \in L$ and there is a point $p \in L$ with $K_p \in T_pL$, then L is totally geodesic and $K_x = \nu(x)\xi_x$ for all $x \in L$ and certain $\nu \in C^\infty(L)$.

Example

Consider the Kruskal space $Q \times_r \mathbb{S}^2$ and

$$L = \{(0, v, x) : v > 0, x \in \mathbb{S}^2\}$$

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For the rigging $\zeta = \partial u$ we have

- $\tau = 0$.
- $C = -vg$.
- $N = \partial u$.
- $\xi \parallel \partial v$.
- $Ric(\xi, N) = K(span(\xi, N)) = 0$.

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- $Ric(\xi, N) = K(span(\xi, N)) = 0$.

$K = v\partial v - u\partial u$ is Killing and causal in L.

Theorem

Let $K \in \mathfrak{X}(M)$ be a timelike conformal vector field with constant conformal factor ρ. Suppose that L is a totally geodesic null hypersurface and ζ is a rigging for L such that

1. $\tau = 0$.
2. $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.
3. $K(span(\xi_x, N_x)) \neq Ric(N_x, \xi_x)$ for all $x \in L$.

Then L can not be compact.

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Example

$(M, g) = \left(\mathbb{S}^1 \times \ldots \times \mathbb{S}^1, dx_1 dx_2 + dx_3^2 + \ldots + dx_n^2\right)$
$K = \partial x_1 - \partial x_2$ is a timelike parallel vector field.

Theorem

Let $K \in \mathfrak{X}(M)$ be a timelike conformal vector field with constant conformal factor ρ. Suppose that L is a totally geodesic null hypersurface and ζ is a rigging for L such that

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$(M, g) = \left(\mathbb{S}^1 \times \ldots \times \mathbb{S}^1, dx_1 dx_2 + dx_3^2 + \ldots + dx_n^2\right)$
$K = \partial x_1 - \partial x_2$ is a timelike parallel vector field.
$L = \{x \in \mathbb{T}^n : x_2 = p\}$ for a fixed $p \in \mathbb{S}^1$ is compact and totally geodesic.
For $\zeta = \partial x_2$ we have

- $\tau = 0$.
- $C = 0$.

Lemma

Let $K \in \mathfrak{X}(M)$ be closed and conformal vector field and fix $p \in M$.

- If K_p is timelike/spacelike then $\mathcal{F}_p$ is a spacelike/timelike totally umbilical hypersurface.
- If K_p is null, then $\mathcal{F}_p$ is a totally umbilical null hypersurface and $Rad(T_x L) = span(K_x)$ for all $x \in \mathcal{F}_p$.

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Lemma

Let $K \in \mathfrak{X}(M)$ be a causal closed and conformal vector field. If K_p is null for some $p \in M$, then $\mathcal{F}_p$ is a totally geodesic null hypersuface.

Question

Under what conditions a null hypersurface is an orthogonal leaf of a closed and conformal vector field?

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Theorem

Let $K \in \mathfrak{X}(M)$ be a causal, closed and conformal vector field. Suppose that L is a null hypersurface with zero null mean curvature and ζ is a rigging for L such that

1. $d\tau = 0$.
2. $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.
3. $0 \leq trace(A^* \circ A)$.
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If $g(\xi, K)Ric(\xi, K) \leq 0$ and there is a point $p \in L$ with $K_p \in T_pL$, then $K_x = \nu(x)\xi_x$ for all $x \in L$ and L is a totally geodesic orthogonal leaf of K.

Theorem

Let $K \in \mathfrak{X}(M)$ be a null parallel vector field and L a null hypersurface with rigging ζ such that

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- $\tau = 0$.
- $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.
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- $K(span(\xi, N)) \leq Ric(\xi, N)$.

If there is $p \in L$ such that $K_p \in T_pL$, then $K_x = \nu \xi_x$ for all $x \in L$ and a nonzero constant $\nu \in \mathbb{R}$ and L is a totally geodesic orthogonal leaf of K.

Theorem

Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field and L a null hypersurface with zero null mean curvature such that $K_x \in T_x L$ for all $x \in L$. Suppose that ζ is a rigging for L such that.

- $d\tau = 0$.
- $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.

If $(n-1)(n-2)\rho^2 \leq Ric(K, K)$ and K_p is null for some point $p \in L$, then $K_x = \nu(x)\xi_x$ for all $x \in L$ and L is a totally geodesic orthogonal leaf of K.

Definition

A null hypersurface has a screen non-degenerate second fundamental form if $B(v, v) \neq 0$ for all $v \in \mathcal{S}$ with $v \neq 0$.

This property does not depend on the chosen rigging.

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Corollary

Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field and L a null hypersurface such that $K_x \in T_x L$ for all $x \in L$. If L is totally umbilic with never vanishing null mean curvature, then $K_x = \nu(x)\xi_x$ for all $x \in L$ and L is an orthogonal leaf of K.

Thank you