On the nature of spacetime singularities

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The singularity theorem by Roger Penrose (1965)

- during the first five decades of Einstein's theory of gravity, singular behavior popped up in many of the physically relevant solutions
- **singularity theorems of Penrose and Hawking**
  - they predict, under a wide range of physically plausible conditions, the existence of **spacetime singularities**, more precisely, they prove that
  - generic spacetimes describing the gravitational collapse of stars (1965) or the expanding universe (1966) are **causal geodesically incomplete**
- **singularity** $\Longleftrightarrow$ **incompleteness**: they are used as synonyms
  - in the case of the well-known solutions, the incomplete causal geodesics **do indeed terminate** on "curvature singularities" (e.g., $R_{abcd}R^{abcd}$ ...)
  - BUT, do we know for sure that the anticipated curvature blow up or any other violent behavior will always occur?
  - NO, NOT REALLY !!! NOT YET !!!
  - $\exists$ inconsistency in the use of notions: incompleteness and singularities
  - outline an argument: a step towards the reduction of this discrepancy

Spacetime as a Cauchy development?

Penrose's strong cosmic censorship conjecture

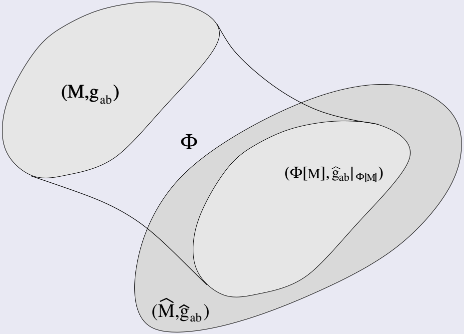
- (M, g_{ab}): M is a smooth paracompact, connected, orientable manifold endowed with a C^∞ metric g_{ab} of Lorentzian signature and time orientation
  - C^∞ only for mathematical conveniences...
  - what is the most suitable differentiability class to be used? (return later)
- tacitly a spacetime is always assumed to represent all the events compatible with the history of the investigated physical system
- a celebrated result by **Choquet-Bruhat and Geroch (1969)** guarantees (assuming smoothness) **the existence of a maximal Cauchy development**
- the situation is more complex: $\exists$ causal geodesically incomplete spacetimes:
  - they **contain as a part** "the maximal" Cauchy development, and
  - they can be continued beyond the Cauchy horizon: no curvature blowup occurs

  (think of max. anal. extensions of Kerr, Taub-NUT spacetimes where the ability to predict the future of data given on an inextendible initial data surface, breaks down)
- **Penrose's strong cosmic censorship conjecture:** the maximal Cauchy development of a **generic** compact or asymptotically flat initial data is never part of a larger spacetime. (if true, $\nexists$ Cauchy Horizon, no extension beyond)

How could we show that something violent is to happen?

- **What do we have by hand?**
  - singularity theorems by **Penrose and Hawking**
  - the existence of maximal Cauchy development **Choquet-Bruhat and Geroch**
  - strong cosmic censorship conjecture by **Penrose**
- **What about the following argument by contradiction?**
  - consider a causal geodesically incomplete spacetime that is the maximal Cauchy development of a **generic** compact or asymptotically flat initial data, i.e. **it is inextendible**
  - assume that nothing violent happens while approaching the **"ideal endpoint"** of any of the incomplete causal geodesics
  - **show** that the considered maximal Cauchy development **can be extended**; *leading to the contradiction:* **it is not maximal, it is extendible**
  - in turn, we get then that **causal geodesic incompleteness of the maximal Cauchy development** must be accompanied by a **singular behaviour**
- **!!!** ...to show that the original spacetime is part of a larger one... we also need some results on *spacetime extensions*: **CJC Clarke** mid 70'; **IR** 1993, 2010

- **Definition:** Consider (M, g_{ab}) and $(\widehat{M}, \widehat{g}_{ab})$ the differential structure of which are at least of class C^X (C^∞?), respectively. (X not specified yet!) A map $\Phi : (M, g_{ab}) \to (\widehat{M}, \widehat{g}_{ab})$ is said to be a $\mathbf{C^X}$**-isometric embedding** if Φ is a C^X-diffeomorphism between M and $\Phi[M] \subset \widehat{M}$ such that it carries the metric g_{ab} into $\widehat{g}_{ab}|_{\Phi[M]}$, i.e. $\Phi^* g_{ab} = \widehat{g}_{ab}|_{\Phi[M]}$.
- **Definition:** $(\widehat{M}, \widehat{g}_{ab})$ is called to be a $\mathbf{C^X}$**-extension** of (M, g_{ab}) if Φ is an isometric embedding such that $\Phi[M]$ is a **proper subset** of $\widehat{M}$



What is the most suitable differentiability class?

- smooth or even C^2 may be too much to be required
- **general relativity is a physical theory:**
  the wider the class of metrics allowed the wider will be the class of physical processes that can be investigated; ??? field equations ???
- **Geroch and Traschen: (1987)** the widest possible class of metrics such that the Riemann, Einstein and Weyl tensors are well-defined as distributions
  - the space of **regular metrics:**
    - g_{ab} locally bounded
    - with locally bounded inverse g^{ab}
    - the weak first derivatives "$\partial_c g_{ab}$" are locally square-integrable
  - C^0 **regular metrics:**
    - if a regular metric g_{ab} is also **continuous** it can be approximated by sequences of smooth metrics $\{^{(i)}g_{ab}\}$ such that the corresponding curvature tensors $\{^{(i)}R_{abc}{}^d\}$ do converge in L_2-norm to the curvature distribution $R_{abc}{}^d$ of the continuous regular metric g_{ab}

The C^0 *G-T regular* metrics are still too rough

- Some of the basic concepts should be well-defined: the class of locally Lipschitz, $C^{0,1}_{loc}$ $(= C^{1-}_{loc})$, metrics is distinguished
- $C^{0,1}_{loc}$ differentiability suffices for many key results of the C^∞ causality theory
  - **global hyperbolicity** makes sense **Chrusciel-Grant** (2012), **Sämann** (2016)
- **global** existence and uniqueness to linear field equations with sources and with g_{ab} of $C^{0,1}_{loc}$ (**Chrusciel-Grant** (2012), **Sanches-Vickers** 2017)
  - $\exists$ **maximal Cauchy development** for the vacuum Einstein equations requires metric with critical Sobolev exponent $s = 2$ (Klainerman-Rodnianski 2001)
- $C^{0,1}_{loc}$ is the weakest condition under which "pointwise" differential geometry is possible (**CJS Clark** 1982)
  - $C^{0,1}_{loc}$-geodesic as uniform accumulation of sequences of smooth ${}^{(i)}g_{ab}$-geodesics
  - weak solutions to Jacobi's equation make sense (a.e.) along timelike geodesics
- physically interesting solutions with $C^{0,1}_{loc}$ regular metrics:
  - gravitational shock waves
  - thin mass shells
  - solutions with pressure free **matter with geodesic flow lines having two- or three-dimensional caustics**

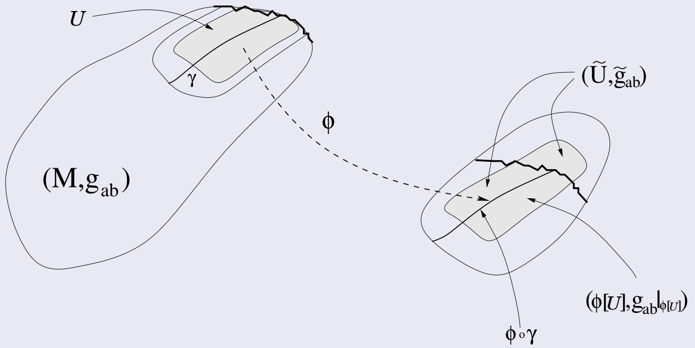
What type of result are we looking for?

- **Theorem:** Consider a timelike geodesically incomplete spacetime (M, g_{ab}) that is the maximal globally hyperbolic development of a generic compact or asymptotically flat initial data and that is inextendible as a Lorentzian manifold within the class of $C^{0,1}_{loc}$ metrics. Then, there exists an incomplete timelike geodesic that terminates on a **parallelly propagated curvature singularity**.

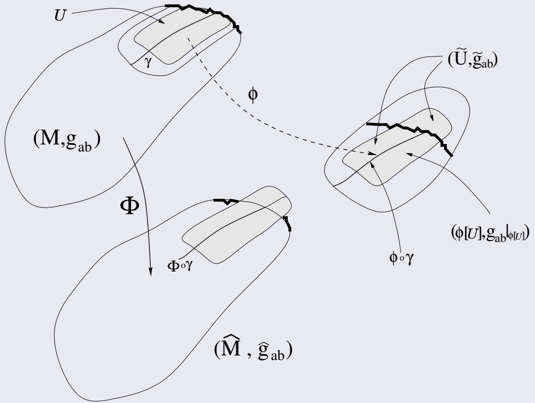
- in what follows the metric will be assumed to be smooth, with the understanding that $C^{0,1}_{loc}$ metrics can be approximated by sequences of C^{∞} metrics
- the papers below provide some hints on results relevant for the smooth setup
  - Rácz, I. (1993): *Spacetime extensions I.*, Journal of Mathematical Physics **34**, 2448-2464
  - Rácz, I. (2010): *Space-time extensions II*, Classical and Quantum Gravity **27**, 155007, Selected by the Editorial Board of Classical and Quantum Gravity as part of the journal's Highlights of 2010, in 2011.

1st step:

- assume that (M, g_{ab}) is a timelike geodesically incomplete, inextendible spacetime and that the **tidal force tensor components are bounded**
- let $\gamma : (t_1, t_*) \to M$ be one of the future incomplete timelike geodesics
  - choose $\mathcal{U}$ be a neighbourhood of a final segment of γ
  - the aim is to find $\widetilde{\mathcal{U}}$ and $\phi : \mathcal{U} \to \widetilde{\mathcal{U}}$ such that $\widetilde{\mathcal{U}}$ is comprised by the union of $\phi[\mathcal{U}]$ and "a neighbourhood of the endpoint of $\phi \circ \gamma$" (not yet constructed **!!!**)



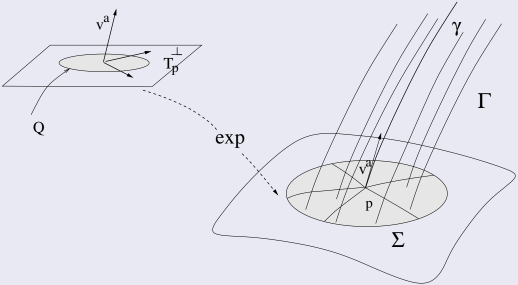
2nd step:



- $(\widehat{M}, \widehat{g}_{ab})$: defined by gluing (M, g_{ab}) and $(\widetilde{\mathcal{U}}, \widetilde{g}_{ab})$ at their "common parts"
  - $\widehat{M}$ is the quotient space $\widehat{M} = (M \cup \widetilde{\mathcal{U}})/\phi$, where $x \in M$ is equivalent to $y \in \widetilde{\mathcal{U}}$ if $\phi(x) = y$; $\widehat{M}$ is Hausdorff with respect to the quotient topology
- the metric $\widehat{g}_{ab}$ on $\widehat{M}$ is determined by g_{ab} and $\widetilde{g}_{ab}$

The construction of $\mathcal{U}$

- to apply Whitney's extension results: cover $\mathcal{U}$ by a single coordinate patch
  - let $\gamma : (t_1, t_*) \to M$ be an incomplete timelike geodesic & $p = \gamma(t_0)$: t is the proper time parameter along γ, with tangent $v^a = (\partial/\partial t)^a$; ($t$ is an affine par.)
  - choose Σ to be the hypersurface generated by spacelike geodesics starting at $p = \gamma(t_0)$ with tangent orthogonal to v^a



- choose an orthonormal frame $\{e^a_{(\mathfrak{a})}\}$, $e^a_{(4)} = v^a$ at p and **extend it onto** Σ
- consider the 3-par. family of timelike geodesics starting at Σ with $v^a = e^a_{(4)}$ ...
- denote by Γ the **"synchronized" 3-par. family of timelike geodesics**
- choose $\mathcal{U}$ to be ruled by members of Γ: extend $\{e^a_{(\mathfrak{a})}\}$ onto $\mathcal{U}$ by par.prop...

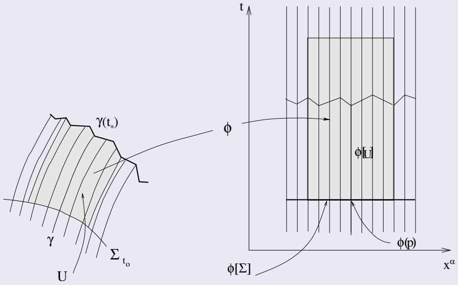
Can Σ be chosen such that $\mathcal{U}$ covers the final segments?

- whenever the tidal force components

$$R_{abcd}e^a_{(\mathfrak{a})}v^b e^c_{(\mathfrak{b})}v^d \qquad\qquad (\text{where } v^a = e^a_{(4)})$$

of the Riemann tensor are bounded in **synchronized, parallelly propagated orthonormal frames** $\{e^a_{(\mathfrak{a})}\}$, along the synchronized 3-parameter congruence Γ (we tacitly assumed that no topological obstruction [alg. special cases] occur)

- Σ can be chosen such that $\mathcal{U}$ is ruled by final segments of members of the timelike geodesic congruence Γ, and such that no conjugate point to Σ occurs



- then Gaussian coordinates $(x^1, x^2, x^3, x^4 = t)$ can be defined on $\mathcal{U}$

How to extend the metric from $\phi[\mathcal{U}] \subset \mathbb{R}^4$?

- **How does one extend a function?**
  - Consider a real-valued C^m-function $\mathcal{F}$ defined on a bounded subset $\mathscr{A}$ of $\mathbb{R}^n$.
  - **How can we tell** whether there exists $\widetilde{\mathcal{F}} \in C^m(\mathbb{R}^n)$ such that $\widetilde{\mathcal{F}} = \mathcal{F}$ on $\mathscr{A}$? [studied by **Hassler Whitney** $\sim$**1930** (see also **C Fefferman (2005)**)]
- **Definition:** a point set $\mathscr{A} \subset \mathbb{R}^n$ is said to possess **the property** $\mathscr{P}$ if there is a positive real number ω such that for any two points x and y of $\mathscr{A}$ can be joined by a curve in $\mathscr{A}$ of length $L \leq \omega \cdot \rho(x, y)$, where $\rho(x, y)$ denotes the Euclidean distance of the points $x, y \in \mathbb{R}^n$.
- **Theorem:** [adopting Whitney's results (1934)] Assume that $\mathscr{A} \subset \mathbb{R}^n$ is bounded and it possesses property $\mathscr{P}$, and let $\mathcal{F}(x^1, ..., x^n)$ be of class $C^{0,1}_{loc}$ in $\mathscr{A}$. Suppose that $\mathcal{F}(x^1, ..., x^n)$ can be defined on the boundary $\partial\mathscr{A}$ of $\mathscr{A}$ such that it is of class $C^{0,1}_{loc}$ on $\overline{\mathscr{A}} = \mathscr{A} \cup \partial\mathscr{A}$. Then there exists an extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ onto $\mathbb{R}^n$ such that $\widetilde{\mathcal{F}}$ is (at least) of class $C^{0,1}_{loc}$.
- as the spacetime is a maximal globally hyperbolic development

  $\Rightarrow$ $\phi[\mathcal{U}] \subset \mathbb{R}^4$ **possesses property** $\mathscr{P}$

We still need to extend the metric from $\phi[\mathcal{U}]$!!!

- in the Gaussian coordinates $(x^1, x^2, x^3, x^4 = t)$ on $\mathcal{U}$: the metric reads as

$$ds^2 = -dt^2 + g_{ij}\, dx^i dx^j$$

- Whitney's theorem $\Rightarrow$ the extendibility of functions defined on bounded subsets of $\mathbb{R}^4$ $\Rightarrow$ the components of the metric g_{ab} can be extended such that its extension $\widetilde{g}_{\alpha\beta}$ is of class $C^{0,1}_{loc}$ if components g_{ij} can be shown to be locally Lipschitz functions on the closure $\overline{\phi[\mathcal{U}]}$ of $\phi[\mathcal{U}]$, which holds
- $\Rightarrow$ if the "t-derivatives" of $g_{ij} = g_{ab}\, E^a_{(i)} E^b_{(j)}$, where $E^a_{(i)} := (\partial/\partial x^i)^a$,

$$\partial_t g_{ij} = v^e \nabla_e [g_{ab}\, E^a_{(i)} E^b_{(j)}] = g_{ab} \left[\left(v^e \nabla_e E^a_{(i)} \right) E^b_{(j)} + E^a_{(i)} \left(v^e \nabla_e E^b_{(j)} \right) \right]$$

  are uniformly bounded along the members of Γ
- it suffices to show that the norms $\|E^a_{(\alpha)}\|$ and $\|v^e \nabla_e E^a_{(\alpha)}\|$ of the coordinate basis fields $E^a_{(i)}$, and also that of $v^e \nabla_e E^a_{(i)}$ are uniformly bounded on $\phi[\mathcal{U}]$
  - where the norm $\|X^a\|$ of a vector field X^a, with respect to a synchronized orthonormal basis field $\{e^a_{(\mathfrak{a})}\}$ and a Lorentzian metric g_{ab}, is defined as

$$\|X^a\| = \sqrt{\sum_{\mathfrak{b}=1}^{4} \left[g_{ab} X^a e^b_{(\mathfrak{b})} \right]^2}$$

- notably, the boundedness can be guaranteed using the Jacobi equation

$$v^e\nabla_e\left(v^f\nabla_f E^a_{(\alpha)}\right) = R_{efg}{}^a\, v^e E^f_{(\alpha)} v^g$$

  and applying our (indirect) assumption guaranteeing the boundedness of the tidal force components of the Riemann tensor, along the members of Γ

- combining all the above partial results $\Rightarrow$
  Theorem: Consider a timelike geodesically incomplete spacetime (M, g_{ab}) that is the maximal globally hyperbolic development of a generic compact or asymptotically flat initial data and that is inextendible within the class of $C^{0,1}_{loc}$ spacetimes. Then, there exists an incomplete timelike geodesic that terminates on a **parallelly propagated curvature singularity**.

- $\Rightarrow$ at least **one of the tidal force tensor components of the Riemann tensor blows up**

Thanks for your attention