Hyperbolicity of GR in null foliations

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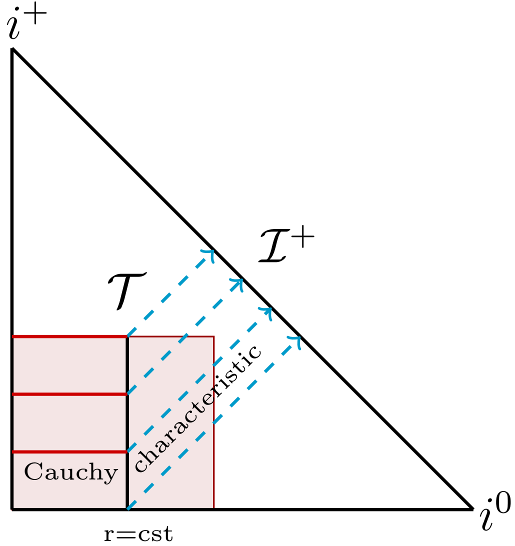
Spanish-Portuguese Relativity Meeting,
Aveiro, September 16, 2021

PhysRevD.102.064035 with D. Hilditch & M. Zilhão

work in progress with N. Bishop, D. Hilditch, D. Pollney & M. Zilhão

Highly accurate gravitational waveform modelling



Cauchy-Characteristic extraction

see e.g. Winicour's 2012 Living Review and references therein

Hyperbolicity

$$\boldsymbol{\mathcal{A}}^t(\mathbf{u}, x^\mu)\,\partial_t\mathbf{u} + \boldsymbol{\mathcal{A}}^p(\mathbf{u}, x^\mu)\,\partial_p\mathbf{u} + \boldsymbol{\mathcal{S}}(\mathbf{u}, x^\mu) = 0\,,$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_q)^T$, is the state vector of the system and

$$\boldsymbol{\mathcal{A}}^\mu = \begin{pmatrix} a^\mu_{11} & \ldots & a^\mu_{1q} \\ \vdots & \ddots & \vdots \\ a^\mu_{q1} & \ldots & a^\mu_{qq} \end{pmatrix}$$

denotes the principal part matrices, with $\det(\boldsymbol{\mathcal{A}}^t) \neq 0\,.$ Construct the

$$\mathbf{P}^s = \left(\boldsymbol{\mathcal{A}}^t\right)^{-1}\boldsymbol{\mathcal{A}}^p\,s_p\,,$$

where s^i is an arbitrary unit spatial vector.

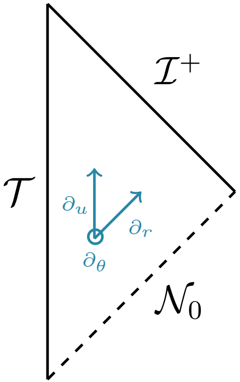
Well-posedness

The PDE problem has a unique solution that depends continuously on the given data in a suitable norm.

- Strongly hyperbolic (SH) $\rightarrow$ **well-posed** IVP in the L^2 norm
- Weakly hyperbolic (WH) $\rightarrow$ **ill-posed** IVP in the L^2 norm

A numerical solution **can converge** to the continuum **only** for well-posed PDE problems.

Bondi-like gauges



- vector basis: $\partial_u\,, \partial_r\,, \partial_\theta\,, \partial_\phi$
- ∂_r is null & $\perp$ to ∂_θ and ∂_ϕ

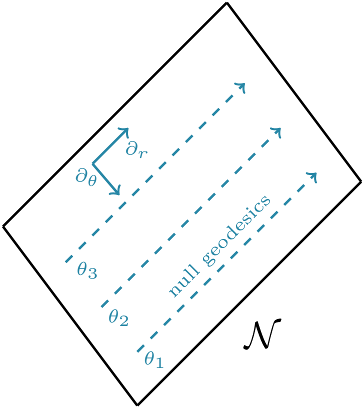
$$g^{\mu\nu} = \begin{pmatrix} 0 & g^{ur} & 0 & 0 \\ g^{ur} & g^{rr} & g^{r\theta} & g^{r\phi} \\ 0 & g^{r\theta} & g^{\theta\theta} & g^{\theta\phi} \\ 0 & g^{r\phi} & g^{\theta\phi} & g^{\phi\phi} \end{pmatrix}$$

Vacuum Einstein's equations:

Evolution system: $R_{rr} = R_{r\theta} = R_{r\phi} = R_{\theta\theta} = R_{\theta\phi} = R_{\phi\phi} = 0$

Bondi, van der Burg & Sachs 1962, Winicour 2013, Cao & He 2013

Weak hyperbolicity of Bondi-like gauges



The principal symbol[1,2]:

$$\mathbf{P}^s = \begin{pmatrix} \boxed{\mathbf{P}_G} & \mathbf{P}_{GC} & \mathbf{P}_{GP} \\ 0 & \mathbf{P}_C & 0 \\ 0 & 0 & \mathbf{P}_P \end{pmatrix}$$

New result[2]: $\mathbf{P}_G$ is non-diagonalizable along θ if ∂_r is $\perp$ to ∂_θ.

GR in Bondi-like gauges $\rightarrow$ WH 2nd order PDE system[3]

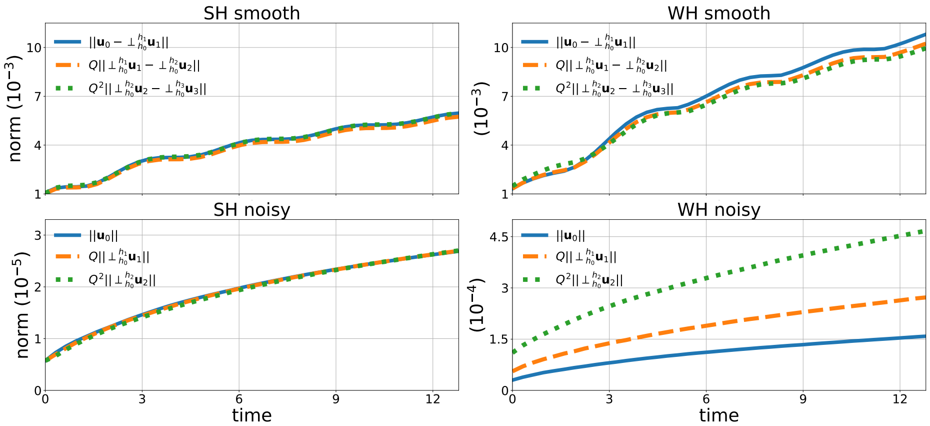
[1] Hilditch & Richter 2016

[2] WIP with Bishop, Hilditch, Pollney & Zilhão

[3] see Ripley 2021 for a symmetric hyperbolic formulation with higher derivatives

Convergence tests in the L^2 norm

- Monitor the numerical error with increasing resolution
- Convergence factor: $Q = 4$ for these tests by construction



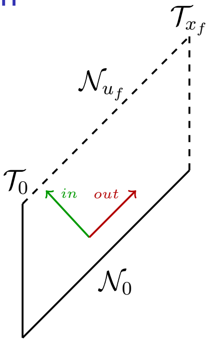
WIP with Bishop, Hilditch Pollney & Zilhão

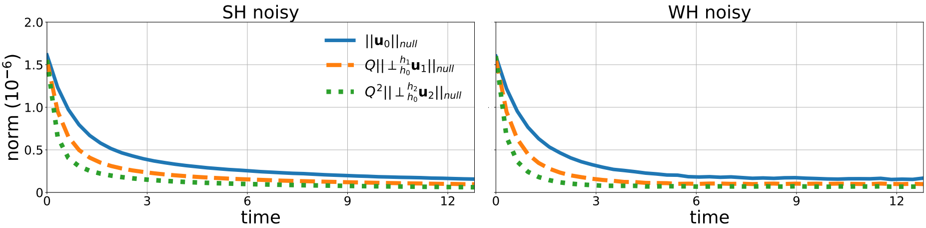
The importance of the norm

Discrete norm[1]:

$$||\mathbf{u}^2|| = ||\mathbf{u}_{\rm in}^2||_{\mathcal{N}_u}^{1/2} + \max_x ||\mathbf{u}_{\rm out}^2||_{\mathcal{T}_x}^{1/2} \, ,$$

where $||...||^{1/2}$ denotes a sum.





WIP with Bishop, Hilditch Pollney & Zilhão

[1] Inspired by the toy models of PhysRevD.102.064035

Summary

- GR in Bondi-like gauges $\rightarrow$ weakly hyperbolic 2nd order PDE system
- Ill-posed characteristic initial boundary value problem in the L^2 norm (other norms?)
- Weak hyperbolicity in numerics $\rightarrow$ high frequency given data

TODO:

- Characteristic GR formulations $\rightarrow$ strongly hyperbolic 2nd order PDE system

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Thank you!