Virial identities and the Gibbons-Hawking-York term

João M. S. Oliveira

September 14, 2021

CENTRA-IST

In collaboration with **A. M. Pombo**, **C. A. R. Herdeiro** and **E. Radu**

arXiv:2109.05027 [gr-qc] - Came out today!

Spanish-Portuguese Relativity Meeting EREP2021

Contents

What is a Virial identity?

- A Virial identity is a relation between energy quantities.

What is a Virial identity?

- A Virial identity is a relation between energy quantities.
- Mathematical identity derived independently of the equations of motion.

What is a Virial identity?

- A Virial identity is a relation between energy quantities.
- Mathematical identity derived independently of the equations of motion.
- Can be used to restrict possibility of solutions and as numerical checks for numerical solutions.

Virial identity in Classical Mechanics

For a stable system of N particles in classical mechanics:

$$\underbrace{\langle T \rangle}_{\text{Kinetic energy}} = \underbrace{-\frac{1}{2}\Sigma_{i=1}^{N}\langle \vec{F}_i \cdot \vec{r}_i \rangle}_{\text{Potential energy}}$$

Virial identity in Classical Mechanics

For a stable system of N particles in classical mechanics:

$$\underbrace{\langle T \rangle}_{\text{Kinetic energy}} = \underbrace{-\frac{1}{2}\Sigma_{i=1}^{N}\langle \vec{F}_i \cdot \vec{r}_i \rangle}_{\text{Potential energy}}$$

- A statistical result!

Virial identity in Classical Mechanics

For a stable system of N particles in classical mechanics:

$$\underbrace{\langle T \rangle}_{\text{Kinetic energy}} = \underbrace{-\frac{1}{2}\Sigma_{i=1}^{N}\langle \vec{F}_i \cdot \vec{r}_i \rangle}_{\text{Potential energy}}$$

- A statistical result!

$$\langle x \rangle = \frac{1}{\Delta t}\int_{t_i}^{t_f} x dt$$

Virial identity in Classical Mechanics

For a stable system of N particles in classical mechanics:

$$\underbrace{\langle T \rangle}_{\text{Kinetic energy}} = \underbrace{-\frac{1}{2}\Sigma_{i=1}^{N}\langle \vec{F}_i \cdot \vec{r}_i \rangle}_{\text{Potential energy}}$$

- A statistical result!

$$\langle x \rangle = \frac{1}{\Delta t}\int_{t_i}^{t_f} x dt$$

We can usually drop the Δt to obtain

$$\int_{t_i}^{t_f}\left(T + \frac{1}{2}\Sigma_{i=1}^{N}\vec{F}_i \cdot \vec{r}_i\right) dt = 0$$

Well known case

Consider

- Conservative forces derivable from a potential U

Well known case

Consider

- Conservative forces derivable from a potential U
- U is a homogeneous function of degree n of the particle's position

Well known case

Consider

- Conservative forces derivable from a potential U
- U is a homogeneous function of degree n of the particle's position

Virial theorem

$$\langle T \rangle = n \frac{\langle U \rangle}{2}$$

Well known case

Consider

- Conservative forces derivable from a potential U
- U is a homogeneous function of degree n of the particle's position

Virial theorem

$$\langle T \rangle = n\frac{\langle U \rangle}{2}$$

- Can also be derived from a variational problem!

Well known case

Consider

- Conservative forces derivable from a potential U
- U is a homogeneous function of degree n of the particle's position

Virial theorem

$$\langle T \rangle = n \frac{\langle U \rangle}{2}$$

- Can also be derived from a variational problem!
- We will discuss this identity in the context of **field theory**.

Contents

Derrick's scaling argument - Scalar field in flat spacetime

Consider the following action

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \left[-\partial_\mu \phi \partial^\mu \phi - U(\phi) \right]$$

Derrick's scaling argument - Scalar field in flat spacetime

Consider the following action

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \left[-\partial_\mu \phi \partial^\mu \phi - U(\phi) \right]$$

Consider

- time independence $\phi = \phi(\vec{r})$

Derrick's scaling argument - Scalar field in flat spacetime

Consider the following action

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \left[-\partial_\mu \phi \partial^\mu \phi - U(\phi) \right]$$

Consider

- time independence $\phi = \phi(\vec{r})$
- the scaling $\vec{r} \to \lambda \vec{r}$

Derrick's scaling argument - Scalar field in flat spacetime

Consider the following action

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \left[-\partial_\mu \phi \partial^\mu \phi - U(\phi) \right]$$

Consider

- time independence $\phi = \phi(\vec{r})$
- the scaling $\vec{r} \to \lambda \vec{r}$
- one-parameter group of scaled functions $\phi_\lambda(\vec{r}) = \phi(\lambda \vec{r})$

Derrick's scaling argument - Scalar field in flat spacetime

Consider the following action

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \left[-\partial_\mu \phi \partial^\mu \phi - U(\phi) \right]$$

Consider

- time independence $\phi = \phi(\vec{r})$
- the scaling $\vec{r} \to \lambda \vec{r}$
- one-parameter group of scaled functions $\phi_\lambda(\vec{r}) = \phi(\lambda \vec{r})$

$$\mathcal{S}_{eff} \to \mathcal{S}^\lambda_{eff} = \frac{1}{4\pi} \int \lambda^3 d^3x \left[-\frac{1}{\lambda^2} (\nabla \phi_\lambda)^2 - U(\phi_\lambda) \right]$$

Derrick's scaling argument - Scalar field in flat spacetime

Consider the following action

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\left[-\partial_\mu\phi\partial^\mu\phi - U(\phi)\right]$$

Consider

- time independence $\phi = \phi(\vec{r})$
- the scaling $\vec{r} \to \lambda\vec{r}$
- one-parameter group of scaled functions $\phi_\lambda(\vec{r}) = \phi(\lambda\vec{r})$

$$\mathcal{S}_{eff} \to \mathcal{S}_{eff}^{\lambda} = \frac{1}{4\pi}\int \lambda^3 d^3x\left[-\frac{1}{\lambda^2}(\nabla\phi_\lambda)^2 - U(\phi_\lambda)\right]$$

$$\left.\frac{\delta\mathcal{S}_{eff}^{\lambda}}{\delta\lambda}\right|_{\lambda=1} = 0$$

Derrick's Virial Identity

$$\int d^3x \left[(\nabla\phi)^2 + 3U(\phi)\right] = 0$$

Derrick's Virial Identity

$$\int d^3x \left[(\nabla\phi)^2 + 3U(\phi)\right] = 0$$

- If $U(\phi) > 0$, both terms are **positive** $\Rightarrow$ no solutions are possible.

Derrick's Virial Identity

$$\int d^3x \left[(\nabla\phi)^2 + 3U(\phi)\right] = 0$$

- If $U(\phi) > 0$, both terms are **positive** $\Rightarrow$ no solutions are possible.
  $\Rightarrow$ Virial identities can then be used to prove **no go theorems**.

Derrick's Virial Identity

$$\int d^3x \left[(\nabla\phi)^2 + 3U(\phi)\right] = 0$$

- If $U(\phi) > 0$, both terms are **positive** $\Rightarrow$ no solutions are possible.
  $\Rightarrow$ Virial identities can then be used to prove **no go theorems**.
- Derrick also showed that no solutions are possible for any $U(\phi)$.

Contents

Heusler-Straumann approach - Asymptotically flat, spherical spacetimes

Consider

Heusler-Straumann approach - Asymptotically flat, spherical spacetimes

Consider

- the action

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_m = \frac{1}{4\pi} \int d^4x \sqrt{-g} \left(\frac{R}{4} + \mathcal{L}_m \right)$$

Heusler-Straumann approach - Asymptotically flat, spherical spacetimes

Consider

- the action

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_m = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} + \mathcal{L}_m\right)$$

- the metric ansatz

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2 \qquad N(r) = 1 - \frac{2m(r)}{r}$$

Heusler-Straumann approach - Asymptotically flat, spherical spacetimes

Consider

- the action

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_m = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} + \mathcal{L}_m\right)$$

- the metric ansatz

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2 \qquad N(r) = 1 - \frac{2m(r)}{r}$$

- Parametrizing functions (σ, m).

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2 \qquad N(r) = 1 - \frac{2m(r)}{r}$$

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2 \qquad N(r) = 1 - \frac{2m(r)}{r}$$

$$\sqrt{-g}R = 4\sigma m' \sin\theta + \frac{d}{dr}(\ldots)$$

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2 \qquad N(r) = 1 - \frac{2m(r)}{r}$$

$$\sqrt{-g}R = 4\sigma m' \sin\theta + \frac{d}{dr}(...)$$

Ignoring total derivative terms...

1D effective action

$$S_{eff} = \int_0^\infty dr \left(\sigma m' + \sigma r^2 \mathcal{L}_m\right)$$

Applying the scaling

- $r \to \lambda r$

Applying the scaling

- $r \to \lambda r$
- $\sigma_\lambda(r) = \sigma(\lambda r), \quad m_\lambda(r) = m(\lambda r)$

Applying the scaling

- $r \to \lambda r$
- $\sigma_\lambda(r) = \sigma(\lambda r), \quad m_\lambda(r) = m(\lambda r)$

$$\frac{\delta S^\lambda_{eff}}{\delta \lambda} = V_m = \int dr \sigma r^2 \frac{\delta}{\delta \lambda}\left(\lambda^3 \mathcal{L}^\lambda_m\right)$$

Gravitational action contribution V_R to the Virial identity **vanishes** for this ansatz.

Example - scalar field in curved spacetime

$$\mathcal{L}_m^{\phi} = -\left(1 - \frac{2m}{r}\right)\phi'^2 - U(\phi)$$

Example - scalar field in curved spacetime

$$\mathcal{L}_m^\phi = -\left(1 - \frac{2m}{r}\right)\phi'^2 - U(\phi)$$

Virial identity - Scalar field

$$\int_0^\infty dr\sigma r^2 \left[\phi'^2 + 3U(\phi)\right] = 0$$

No go theorem still valid in curved spherical spacetime!

Black Hole spacetimes

For Black hole spacetimes...

Black Hole spacetimes

For Black hole spacetimes...

- The integration range is now $[r_H, +\infty[$.

Black Hole spacetimes

For Black hole spacetimes...

- The integration range is now $[r_H, +\infty[$.
  $\Rightarrow r \to r_\lambda = r_H + \lambda(r - r_H)$.

Black Hole spacetimes

For Black hole spacetimes...

- The integration range is now $[r_H, +\infty[$.
  $\Rightarrow r \to r_\lambda = r_H + \lambda(r - r_H)$.
- $\sigma_\lambda(r) = \sigma(r_\lambda), \quad m_\lambda(r) = m(r_\lambda), \quad \phi_\lambda(r) = \phi(r_\lambda)$

Black Hole spacetimes

For Black hole spacetimes...

- The integration range is now $[r_H, +\infty[$.
  $\Rightarrow r \to r_\lambda = r_H + \lambda(r - r_H)$.
- $\sigma_\lambda(r) = \sigma(r_\lambda), \quad m_\lambda(r) = m(r_\lambda), \quad \phi_\lambda(r) = \phi(r_\lambda)$

Virial identity - Scalar field, BH spacetime

$$\int_0^\infty dr\sigma r^2 \left[\left(1 + \frac{2r_H}{r} \left(\frac{m}{r} - 1 \right) \right) \phi'^2 + \left(3 - \frac{2r_H}{r} \right) U(\phi) \right] = 0$$

Both coefficients still positive $\Rightarrow$ Theorem still holds!

Contents

An inconsistency - the total derivative term

Consider a functional $f[\sigma(r), m(r), r]$ such that

$$\mathcal{S}'_{eff} = \mathcal{S}_{eff} + \int dr \frac{d}{dr} f[\sigma(r), m(r), r]$$

An inconsistency - the total derivative term

Consider a functional $f[\sigma(r), m(r), r]$ such that

$$\mathcal{S}'_{eff} = \mathcal{S}_{eff} + \int dr \frac{d}{dr} f[\sigma(r), m(r), r]$$

- The equations of motion for $\mathcal{S}'$ and $\mathcal{S}$ are the same.

An inconsistency - the total derivative term

Consider a functional $f[\sigma(r), m(r), r]$ such that

$$\mathcal{S}'_{eff} = \mathcal{S}_{eff} + \int dr \frac{d}{dr} f[\sigma(r), m(r), r]$$

- The equations of motion for $\mathcal{S}'$ and $\mathcal{S}$ are the same.
- However the **Virial identities are not**

$$\left.\frac{\delta S^{\lambda}_{eff}}{\delta \lambda}\right|_{\lambda=1} + \left[\left.\frac{\delta}{\delta \lambda} f[\sigma_\lambda(r), m_\lambda(r), \lambda r]\right|_{\lambda=1}\right]_0^\infty = 0$$

What is the correct surface term that we should take into account?

Another inconsistency - the ansatz

- Heusler-Straumann approach does not work for other ansatze.

Another inconsistency - the ansatz

- Heusler-Straumann approach does not work for other ansatze.

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

- (σ, N) as parametrising functions

Another inconsistency - the ansatz

- Heusler-Straumann approach does not work for other ansatze.

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

- (σ, N) as parametrising functions
- Ignoring total derivative terms V_{Td}, still results in a contribution from the **gravitational term** V_R

Another inconsistency - the ansatz

- Heusler-Straumann approach does not work for other ansatze.

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

- (σ, N) as parametrising functions
- Ignoring total derivative terms V_{Td}, still results in a contribution from the **gravitational term** V_R
- V_R does **not vanish** for Schwarzschild spacetime

Another inconsistency - the ansatz

- Heusler-Straumann approach does not work for other ansatze.

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

- (σ, N) as parametrising functions
- Ignoring total derivative terms V_{Td}, still results in a contribution from the **gravitational term** V_R
- V_R does **not vanish** for Schwarzschild spacetime
- Total derivative (surface) term V_{Td} is **required**.

The Gibbons-Hawking-York (GHY) term

$$\mathcal{S} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-\gamma}(K - K_0)$$

The Gibbons-Hawking-York (GHY) term

$$\mathcal{S} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-\gamma}(K - K_0)$$

- $\gamma_{\mu\nu}$ - 3-metric of the boundary ∂M

The Gibbons-Hawking-York (GHY) term

$$\mathcal{S} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-\gamma}(K - K_0)$$

- $\gamma_{\mu\nu}$ - 3-metric of the boundary ∂M
- K - Extrinsic curvature of ∂M

The Gibbons-Hawking-York (GHY) term

$$\mathcal{S} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} R + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{-\gamma} (K - K_0)$$

- $\gamma_{\mu\nu}$ - 3-metric of the boundary ∂M
- K - Extrinsic curvature of ∂M
- K_0 - Extrinsic curvature of ∂M imbedded in flat spacetime

The Gibbons-Hawking-York (GHY) term

$$\mathcal{S} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-\gamma}(K - K_0)$$

- $\gamma_{\mu\nu}$ - 3-metric of the boundary ∂M
- K - Extrinsic curvature of ∂M
- K_0 - Extrinsic curvature of ∂M imbedded in flat spacetime
- Necessary term to have a **well posed variational principle** in a manifold M with a boundary ∂M

The Gibbons-Hawking-York (GHY) term

$$\mathcal{S} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{-\gamma}(K - K_0)$$

- $\gamma_{\mu\nu}$ - 3-metric of the boundary ∂M
- K - Extrinsic curvature of ∂M
- K_0 - Extrinsic curvature of ∂M imbedded in flat spacetime
- Necessary term to have a **well posed variational principle** in a manifold M with a boundary ∂M

- We have **three** gravitational Virial contributions

$$\underbrace{V_R + \underbrace{V_{Td} + V_{GHY}}_{\text{Surface terms}}}_{\text{Gravitational contribution}} + \underbrace{V_m}_{\text{Matter}} = 0$$

GHY Virial contribution

$$V_{GHY} = 2\left[\frac{\delta}{\delta\lambda}\left(\sqrt{-\gamma_\lambda}(K_\lambda - K_{0\lambda})\right)\Big|_{\lambda=1}\right]_{\partial M}$$

- Along with the total derivative contribution V_{Td}, it **removes the second derivative terms** from the action.

GHY contribution for the (σ, m) ansatz

For the (σ, m) ansatz

$$V_S = V_{GHY} + V_{Td} = 4\sigma\left(1 - \frac{r-m}{\sqrt{r(r-2m)}}\right)(r - r_H)$$

GHY contribution for the (σ, m) ansatz

For the (σ, m) ansatz

$$V_S = V_{GHY} + V_{Td} = 4\sigma \left(1 - \frac{r-m}{\sqrt{r(r-2m)}}\right)(r - r_H)$$

- Vanishes for most solutions! $\Rightarrow$ reason why the H-S approach works

GHY contribution for the (σ, m) ansatz

For the (σ, m) ansatz

$$V_S = V_{GHY} + V_{Td} = 4\sigma\left(1 - \frac{r-m}{\sqrt{r(r-2m)}}\right)(r - r_H)$$

- Vanishes for most solutions! $\Rightarrow$ reason why the H-S approach works
- Other ansatze can have non-vanishing V_S

The (σ, N) ansatz

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

The (σ, N) ansatz

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

- σ and N parametrising functions.

The (σ, N) ansatz

$$ds^2 = -\sigma^2(r)N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2$$

- σ and N parametrising functions.
- $V_{GHY} = 0$ for **Schwarzschild spacetime** $\Rightarrow$ requires only the V_R and V_{Td} terms

$$\Rightarrow V_R + V_{Td} = 0$$

The necessity of the GHY term - Reissner-Nordström spacetime

What about **Reissner-Nordström** spacetime in the (σ, N) ansatz?

The necessity of the GHY term - Reissner-Nordström spacetime

What about **Reissner-Nordström** spacetime in the (σ, N) ansatz?

$$\mathcal{L}_m = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow V_{EM} = 0$$

The necessity of the GHY term - Reissner-Nordström spacetime

What about **Reissner-Nordström** spacetime in the (σ, N) ansatz?

$$\mathcal{L}_m = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow V_{EM} = 0$$

- All gravitational contributions are **non-vanishing**

$$\Rightarrow V_R + V_{Td} + V_{GHY} = 0$$

The necessity of the GHY term - Reissner-Nordström spacetime

What about **Reissner-Nordström** spacetime in the (σ, N) ansatz?

$$\mathcal{L}_m = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Rightarrow V_{EM} = 0$$

- All gravitational contributions are **non-vanishing**

$$\Rightarrow V_R + V_{Td} + V_{GHY} = 0$$

- This shows how both the total derivative term V_{Td} and the GHY contribution V_{GHY} are **necessary**.

Contents

Summary and Implications

- We realized that the GHY term is **essential** to the complete and correct derivation of the Virial identity.

Summary and Implications

- We realized that the GHY term is **essential** to the complete and correct derivation of the Virial identity.
- This more complete derivation allows us to consider other **different ansatze**.

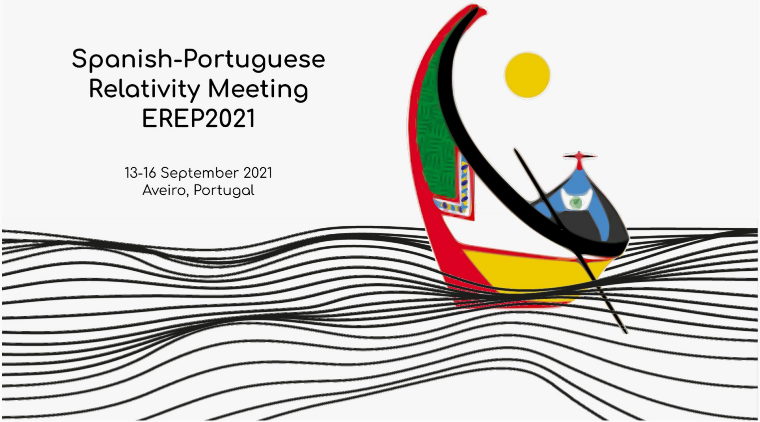
Summary and Implications

- We realized that the GHY term is **essential** to the complete and correct derivation of the Virial identity.
- This more complete derivation allows us to consider other **different ansatze**.
- Allows the calculation of virial identities for other kinds of **symmetries** and **higher dimensional effective actions**, like stationary and axisymmetric spacetimes.

Summary and Implications

- We realized that the GHY term is **essential** to the complete and correct derivation of the Virial identity.
- This more complete derivation allows us to consider other **different ansatze**.
- Allows the calculation of virial identities for other kinds of **symmetries** and **higher dimensional effective actions**, like stationary and axisymmetric spacetimes.
- All this and more in a future companion paper.

To be continued...



Thank You!

arXiv:2109.05027 [gr-qc]

Einstein-Hilbert integrand

$$\frac{\sqrt{-g}}{\sin\theta}R = 2\sigma\left(1 - rN' - N\right) - \frac{d}{dr}\left(2r^2\sigma' N + r^2\sigma N'\right)$$
$$= 4\sigma m' + 2\frac{d}{dr}\left[r\sigma'(2m - r) + \sigma(m' r - m)\right]$$

GHY integrand

$$\frac{\sqrt{-\gamma}}{\sin\theta}(K - K_0) = \frac{r^2}{2}\sigma N' + 2r\sigma(N - \sqrt{N}) + r^2\sigma' N$$
$$= r\sigma'(r - 2m) - \sigma\left(m' r + 2r\sqrt{1 - \frac{2m}{r}} - 2r + 3m\right)$$

Virial - vacuum GR $(\sigma - N)$ ansatz

$$2\int_{r_H}^{\infty} \sigma \left[N - 1 + (r - r_H)N'\right] dr = \left[4\sigma(N - \sqrt{N})(r - r_H)\right]_{r_H}^{+\infty}$$

Virial - vacuum GR (σ, m) ansatz

$$\left[-4\sigma\left(\frac{r-m}{\sqrt{r^2 - 2mr}} - 1\right)(r - r_H)\right]_{r_H}^{+\infty} = 0$$