Gravitomagnetism in the Lewis cylindrical metrics

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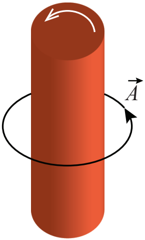

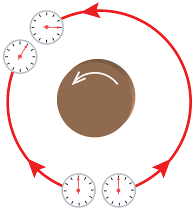

Lewis metrics

Describe generically the exterior gravitational field produced by infinitely long cylinders

$$ds^2 = -fdt^2 + 2kdtd\phi + r^{(n^2-1)/2}(dr^2 + dz^2) + ld\phi^2$$

$$f = ar^{1-n} - \frac{c^2 r^{n+1}}{n^2 a}; \qquad k = -Cf; \qquad l = \frac{r^2}{f} - C^2 f; \qquad C = \frac{cr^{n+1}}{naf} + b$$

Two classes:

- Lewis class (complex parameters)
- Weyl class (all parameters real)
  - static cylinders (Levi-Civita metric)
  - rotating cylinders

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Two classes:

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  - rotating cylinders

Field of static and rotating cylinders of the Weyl class

- both *locally* static; *locally* indistinguishable
- but known to globally differ (matching to the source)
  - significance was unclear (physical and geometrical)
  - in which physical effects the rotation imprints itself?

Gravitomagnetism

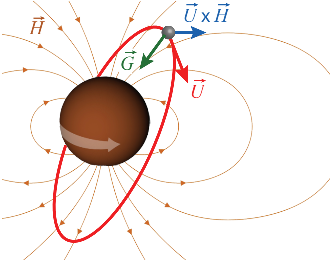
Stationary spacetime: $ds^2 = -e^{2\Phi}(dt - \mathcal{A}_i dx^i)^2 + h_{ij} dx^i dx^j$

- $h_{ij} \equiv$ spatial metric (Einstein's light signaling procedure)
- $\Phi \equiv$ "gravitoelectric" potential;
- $\vec{\mathcal{A}} \equiv$ "gravitomagnetic" vector potential;

Spatial components of *exact* geodesic equation:

$$\frac{\tilde{D}\vec{U}}{d\tau} = \gamma\left[\gamma\vec{G} + \vec{U} \times \vec{H}\right] \quad \left(\text{analogous to Lorentz force } \frac{D\vec{U}}{d\tau} = \frac{q}{m}\left[\gamma\vec{E} + \vec{U} \times \vec{B}\right]\right)$$

- $\vec{G} = -\tilde{\nabla}\Phi \equiv$ gravitoelectric field;
- $\vec{H} = e^{\Phi}\tilde{\nabla} \times \vec{\mathcal{A}} \equiv$ gravitomagnetic field;
- $\tilde{\nabla} \equiv$ Levi-Civita connection of h_{ij}
- $\tilde{D}U^i/d\tau = dU^i/d\tau + \Gamma(h)^i_{jk} U^j U^k$
  (covariant derivative with respect to h_{ij})

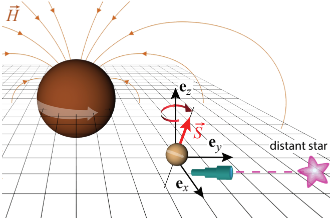


Gravitomagnetism

Stationary spacetime: $ds^2 = -e^{2\Phi}(dt - \mathcal{A}_i dx^i)^2 + h_{ij} dx^i dx^j$

($\Phi \equiv$ "gravitoelectric" potential; $\vec{\mathcal{A}} \equiv$ "gravitomagnetic" potential
$h_{ij} \equiv$ spatial metric)

- Gyroscope precession: $\dfrac{d\vec{S}}{d\tau} = \dfrac{1}{2}\vec{S} \times \vec{H}$
  - $\vec{H} = e^{\Phi}\tilde{\nabla} \times \vec{\mathcal{A}} \equiv$ gravitomagnetic field;
  - analogous to precession of magnetic dipole ($D\vec{S}/d\tau = \vec{\mu} \times \vec{B}$)



- Force on gyroscope: $\dfrac{DP^\alpha}{d\tau} = -\mathbb{H}^{\beta\alpha} S_\beta$
  - $\mathbb{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu} U^\mu U^\nu \equiv$ gravitomagnetic tidal tensor
  - analogous to force on magnetic dipole $DP^\alpha/d\tau = B^{\beta\alpha}\mu_\beta$; $B_{\alpha\beta} = \star F_{\alpha\mu;\beta} U^\mu \equiv$ magnetic tidal tensor
  - $\mathbb{H}_{ij} = -\frac{1}{2}\left[\tilde{\nabla}_j H_i + (\vec{G} \cdot \vec{H}) h_{ij} - 2 G_j H_i\right]$

Levels of Magnetism		Levels of Gravitomagnetism	
Field	Physical effect	Field	Physical effect
$\vec{A}$ (magnetic v. potential)	• Aharonov -Bohm effect (quantum theory)	$\vec{\mathcal{A}}$ (gravitomag. v. potential)	• Sagnac effect • part of GM clock effect
$\vec{B}$ $(\nabla \times \vec{A})$	• magnetic force $q\vec{U} \times \vec{B}$ • dipole precess. $\frac{D\vec{S}}{d\tau} = \vec{\mu} \times \vec{B}$ • magnetic clock effect	$\vec{H}$ $(e^{\Phi}\nabla \times \vec{\mathcal{A}})$	• gravitomag. force $m\gamma\vec{U} \times \vec{H}$ • gyroscope precess. $\frac{d\vec{S}}{d\tau} = \frac{1}{2}\vec{S} \times \vec{H}$ • part of GM clock effect
$B_{\alpha\beta}$ $(\sim \partial_i\partial_j A_k)$	• Force on mag. dipole $\frac{DP^{\alpha}}{d\tau} = B^{\beta\alpha}\mu_{\beta}$	$\mathbb{H}_{\alpha\beta}$ $(\sim \partial_i\partial_j \mathcal{A}_k)$	• Force on gyroscope $\frac{DP^{\alpha}}{d\tau} = -\mathbb{H}^{\beta\alpha}S_{\beta}$

EM field of infinitely long rotating charged cylinder

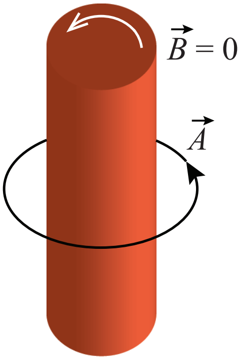
$\varphi = -2\lambda \ln(r)$ (Electric potential)

$\vec{A} = \frac{\mathfrak{m}}{r^2}\partial_\phi$ (Magnetic vector potential)

$\boldsymbol{A} = \mathfrak{m}d\phi$ (Magnetic potential 1-form; constant components in coord. basis)

$\vec{E} = -\nabla\varphi = \frac{2\lambda}{r}\partial_r$ (Electric field)

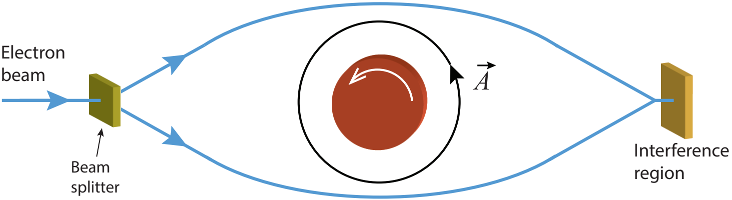
$\vec{B} = \nabla \times \vec{A} = 0$ (Magnetic field)



- Same φ, $\vec{E}$ and $\vec{B} = 0$ as for a static cylinder
  - only differ in $\vec{A}$ ($= 0$ for static cylinder)
  - *classically*, curl-free $\vec{A}$ is gauge; physics depend only on $\vec{E}$ and $\vec{B}$
  - field of static and rotating cylinders indistinguishable *classically*

Aharonov-Bohm effect

Beam of electrons split and passing around a rotating charged cylinder (avoiding it)



- $\vec{A}$ induces a phase shift in electron's wave function ($\varphi = q/\hbar \int_C \boldsymbol{A} \equiv q/\hbar \int_C \vec{A} \cdot d\vec{l}$)
- Phase difference between the two paths $\Delta\varphi = \dfrac{q}{\hbar} \oint_C \boldsymbol{A} = \dfrac{2\pi q}{\hbar} A_\phi$

Lewis metrics — Weyl class

$$ds^2 = -fdt^2 + 2kdtd\phi + r^{(n^2-1)/2}(dr^2 + dz^2) + ld\phi^2$$

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- Φ complicated
- $\boldsymbol{\mathcal{A}} = \mathcal{A}_\phi \mathrm{d}\phi$; $\mathcal{A}_\phi$ complicated
- $\vec{H} \neq 0$, $\mathbb{H}_{\alpha\beta} \neq 0$
- all unlike EM analogue *in inertial rest frame*
- resembles EM analogue in a rotating frame

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- all unlike EM analogue *in inertial rest frame*
- resembles EM analogue in a rotating frame
- ∂_t ceases to be time-like for $r^{2n} > a^2 n^2 / c^2$
  $\Rightarrow$ no observers at rest are possible past that r
  - resembles a rigidly rotating frame in flat spacetime
- could the metric, as usually given in the literature, be actually written in a rotating coordinate system?

Lewis metrics — Weyl class

From the invariants of the Riemann (= Weyl) tensor

- quadratic $\begin{cases} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{1}{4}(n^2-1)^2(3+n^2)r^{-3-n^2} \neq 0 \\ \star R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 0 \end{cases}$

- cubic $\begin{cases} R^{\alpha\beta}{}_{\lambda\mu}R^{\lambda\mu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} = -\frac{3}{16}(n^2-1)^4 r^{-3(n^2+3)/2} \\ R^{\alpha\beta}{}_{\lambda\mu}R^{\lambda\mu}{}_{\rho\sigma}\star R^{\rho\sigma}{}_{\alpha\beta} = 0 \end{cases}$

we know it is a purely "electric" Petrov type I spacetime

- at each point, an observer exists measuring $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu}U^\mu U^\nu = 0$

Lewis metrics — "canonical" form of Weyl class

Observers measuring $\mathbb{H}_{\alpha\beta} = 0$ have 4-velocity $U^\alpha = U^t(\delta^\alpha_t + \Omega\delta^\alpha_\phi)$, with *constant* angular velocity

$$\Omega = \frac{c}{n - bc} \quad \text{or} \quad \Omega = -\frac{1}{b} \qquad \text{(redundancy in original parameters)}$$

- *Rigid* observer congruence!
- Transformation $\bar{\phi} = \phi - \Omega t$, at constant Ω, yields a coordinate system where they are at rest; redefining parameters:

$$ds^2 = -\frac{r^{4\lambda_\mathrm{m}}}{\alpha}\left[dt - \frac{j}{\lambda_\mathrm{m} - 1/4}d\bar{\phi}\right]^2 + r^{4\lambda_\mathrm{m}(2\lambda_\mathrm{m}-1)}(dr^2 + dz^2) + \alpha r^{2(1-2\lambda_\mathrm{m})}d\bar{\phi}^2$$

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- Only 3 parameters (originally 4), with clear physical meaning
  - $\lambda_\mathrm{m} \equiv$ Komar mass per unit $z-$ length
  - $j \equiv$ Komar angular momentum per unit $z-$ length
  - α governs the angle deficit

Lewis metrics — "canonical" form of Weyl class

$$ds^2 = -e^{2\Phi}(dt - \mathcal{A}_{\bar\phi} d\bar\phi)^2 + h_{ij} dx^i dx^j$$

$$e^{2\Phi} = \frac{1}{\alpha} r^{4\lambda_\mathrm{m}} \qquad \mathcal{A}_{\bar\phi} = \frac{j}{\lambda_\mathrm{m} - 1/4}$$

$$h_{rr} = h_{zz} = r^{(n^2-1)/2} \qquad h_{\bar\phi\bar\phi} = \alpha r^{2(1-2\lambda_\mathrm{m})}$$

$\Phi = 2\lambda_\mathrm{m} \ln(r) + K$ (gravitoelectric potential)

$G_i = -\Phi_{,i} = -\frac{2\lambda_\mathrm{m}}{r}\delta^r_i$ (gravitoelectric field)

$\boldsymbol{\mathcal{A}} = \mathcal{A}_\phi \mathrm{d}\phi$; $\mathcal{A}_{\bar\phi}$ *constant* (gravitomagnetic potential 1-form)

$H^i = e^\Phi \epsilon^{ijk} \partial_j \mathcal{A}_k = 0$ (gravitomagnetic field)

Exactly mirrors electromagnetic analogue

- Φ and G_i match exactly φ and E_i, identifying $\lambda_\mathrm{m} \leftrightarrow -\lambda$
- $\vec{H}$ and $\mathbb{H}_{\alpha\beta}$ vanish (like $\vec{B} = B_{\alpha\beta} = 0$)
- $\boldsymbol{\mathcal{A}}$, like $\boldsymbol{A}$, is azimuthal and *irrotational* (i.e., closed form, $\mathrm{d}\boldsymbol{\mathcal{A}} = 0$)

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The Killing vector ∂_t is time-like *everywhere*

- observers at rest possible everywhere

$\vec{G} \overset{r\to\infty}{\to} \vec{0} \Rightarrow$ frame asymptotically inertial

- frame fixed to the "distant stars"

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- canonical form of the Weyl class Lewis metric

“Canonical” form of Weyl class — notable limits

$$ds^2 = -\frac{r^{4\lambda_m}}{\alpha}\left[dt - \frac{j}{\lambda_m - 1/4}d\bar{\phi}\right]^2 + r^{4\lambda_m(2\lambda_m-1)}(dr^2+dz^2) + \alpha r^{2(1-2\lambda_m)}d\bar{\phi}^2$$

- $\lambda_m \to 0 \;\Rightarrow$ spinning cosmic string:

$$ds^2 = -\frac{1}{\alpha}\left[dt + 4j d\bar{\phi}\right]^2 + dr^2 + dz^2 + \alpha r^2 d\bar{\phi}^2$$

- $j \to 0 \;\Rightarrow$ static Levi-Civita cylinder:

$$ds^2 = -\frac{r^{4\lambda_m}}{\alpha}dt^2 + r^{4\lambda_m(2\lambda_m-1)}(dr^2+dz^2) + \alpha r^{2(1-2\lambda_m)}d\bar{\phi}^2$$

“Canonical” form of Weyl class vs Levi-Civita

$$ds^2 = -e^{2\Phi}(dt - \mathcal{A}_{\bar{\phi}} d\bar{\phi})^2 + h_{ij} dx^i dx^j$$

$$e^{2\Phi} = \frac{1}{\alpha} r^{4\lambda_{\rm m}} \qquad \mathcal{A}_{\bar{\phi}} = \frac{j}{\lambda_{\rm m} - 1/4}$$

$\Phi = 2\lambda_{\rm m} \ln(r) + K$ (gravitoelectric potential)

$G_i = -\Phi_{,i} = -\frac{2\lambda_{\rm m}}{r}\delta_i^r$ (gravitoelectric field)

$\boldsymbol{\mathcal{A}} = \mathcal{A}_\phi \mathrm{d}\phi$; $\mathcal{A}_{\bar{\phi}}$ *constant* (gravitomagnetic potential 1-form)

$H^i = e^{\Phi}\epsilon^{ijk}\partial_j \mathcal{A}_k = 0$ (gravitomagnetic field)

Φ, $\vec{G}$, $\vec{H}$, h_{ij}, match those of the static Levi-Civita cylinder

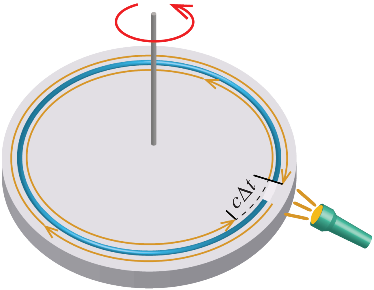
- all inertial and tidal fields/forces are the same

Only differ in $\boldsymbol{\mathcal{A}}$ ($= 0$ for static cylinder)

- Again, like in electromagnetic analogue, with $\boldsymbol{A}$

Sagnac effect

Light beams propagating in opposite directions along optical fiber loop



Loop attached to a rotating platform (turntable) in flat spacetime

- Take different times to complete the loop; co-rotating beam takes longer
- co-rotating beam undergoes a longer path, because arrival point is "running away" from the beam during the trip
- counter-rotating one undergoes a shorter path, as arrival point is approaching it during the trip.

Measures the apparatus' absolute rotation with respect to inertial frame

Sagnac effect

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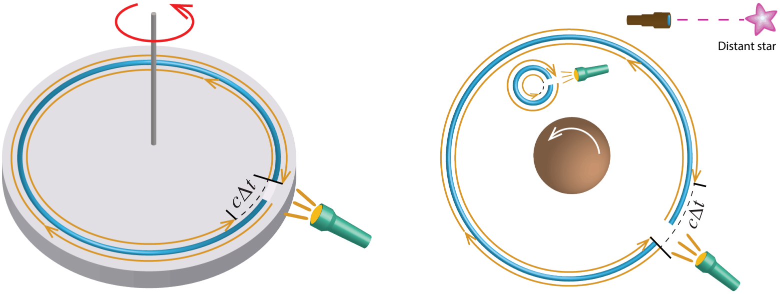


Loops fixed with respect to the distant stars, and placed close to a spinning body

- Again beams take different times to complete the loop

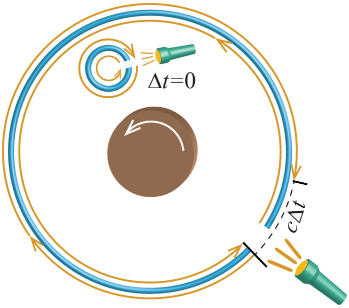
  $\Rightarrow$ now assigned to *frame-dragging*

In both cases: $ds^2 = -e^{2\Phi}(dt - \mathcal{A}_\phi d\phi)^2 + h_{ij} dx^i dx^j$

- difference in arrival times: $\Delta t = 2 \oint_C \boldsymbol{\mathcal{A}} \equiv 2 \oint_C \mathcal{A}_i dx^i$

Sagnac effect around cylinder of the Weyl class

Loops at rest in star fixed ("canonical") coordinates



- $\Delta t = 2 \oint_C \boldsymbol{\mathcal{A}} \equiv 2 \oint_C \mathcal{A}_i dx^i$
- $\mathrm{d}\boldsymbol{\mathcal{A}} = 0 \quad \Leftrightarrow \quad \nabla \times \vec{\mathcal{A}} = 0$ (*closed* form)

By Stokes theorem:

- $\Delta t = 0$ for any loop **not enclosing** the cylinder
- $\Delta t = 4\pi \mathcal{A}_\phi$ the same for all loops **enclosing** the cylinder

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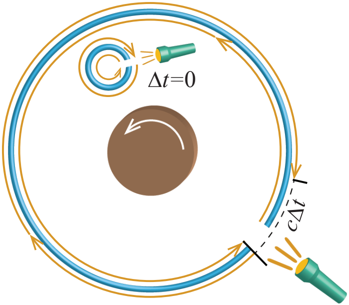

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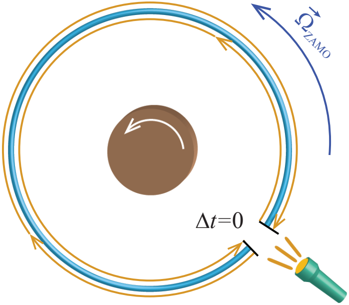
By Stokes theorem:

- $\Delta t = 0$ for any loop **not enclosing** the cylinder
- $\Delta t = 4\pi\mathcal{A}_\phi$ the same for all loops **enclosing** the cylinder

- Mirrors the Aharonov-Bohm effect around spinning charged cylinders
  - $\mathrm{d}\boldsymbol{A} = 0 \implies \Delta\varphi = q/\hbar \oint_C \boldsymbol{A} = 2\pi q/\hbar A_\phi$ path-independent
  - Sagnac phase difference: $\Delta\varphi = 2\pi E/\hbar\mathcal{A}_\phi$ (formally analogous)

Sagnac effect around cylinder of the Weyl class



Distinction can be made without use of a specific frame

- but not with a single loop
- Sagnac effect in a loop can be made to vanish by spinning it

Effect vanishes in circular loops with *zero angular momentum*

- those rotating with angular velocity

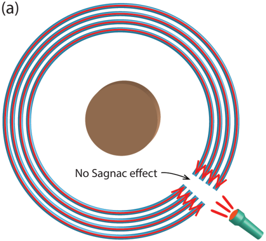
$$\Omega_{\mathrm{ZAMO}}(r) = -\frac{g_{0\phi}}{g_{\phi\phi}} = -\left[\frac{j}{1/4-\lambda_{\mathrm{m}}} - \frac{1/4-\lambda_{\mathrm{m}}}{j}\alpha^2 r^{2(1-4\lambda_{\mathrm{m}})}\right]^{-1}$$

(r- dependent!)

Sagnac effect around cylinder of the Weyl class

Coil of optical fiber loops

(a)



- In static (Levi-Civita) cylinder, Sagnac effect can be made to vanish simultaneously in every loop
  (namely, when the coil is at rest relative to distant stars)

Sagnac effect around cylinder of the Weyl class

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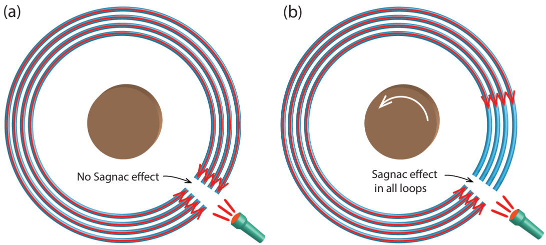

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Sagnac effect around cylinder of the Weyl class

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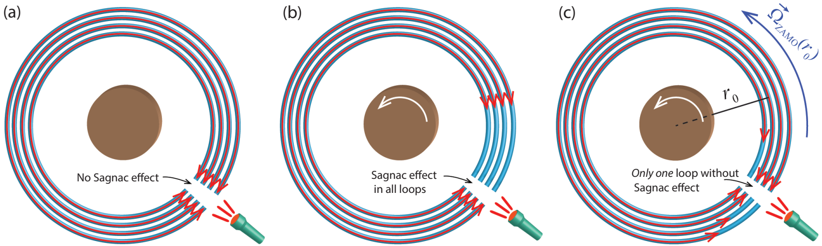
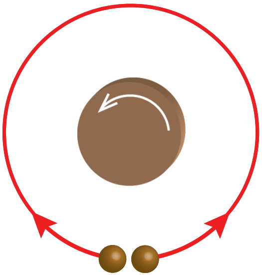


- In static (Levi-Civita) cylinder, Sagnac effect can be made to vanish simultaneously in every loop
(namely, when the coil is at rest relative to distant stars)
- In rotating cylinder, and coil fixed to the distant stars, Sagnac effect arises in every loop
- Spinning the coil with angular velocity $\Omega_{\rm ZAMO}(r_0)$ makes the effect vanish at a loop of radius r_0;
  - but on all other loops a Sagnac effect will arise

Gravitomagnetic clock effect



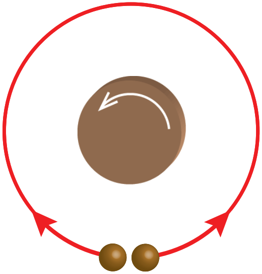
Around a spinning body, the periods of co- and counter-rotating geodesics differs:

$$\Delta t_{\rm geo} = \underbrace{4\pi \mathcal{A}_\phi}_{\rm Sagnac} + 2\pi \frac{\sqrt{h} H^z}{G_r e^\Phi} \; ,$$

$\Delta t_{\rm geo}$ consists of two terms

- one equaling the Sagnac time delay $4\pi \mathcal{A}_\phi$
- plus one due to the gravitomagnetic force $\gamma \vec{U} \times \vec{H}$
  - repulsive (attractive) for co-(counter) rotating geodesics
  - analogous to magnetic force produced by spinning body

Gravitomagnetic clock effect



Around a spinning body, the periods of co- and counter-rotating geodesics differs:

$$\Delta t_{\rm geo} = \underbrace{4\pi\mathcal{A}_\phi}_{\rm Sagnac} + 2\pi\frac{\sqrt{h}H^z}{G_r e^\Phi} ,$$

$\Delta t_{\rm geo}$ consists of two terms

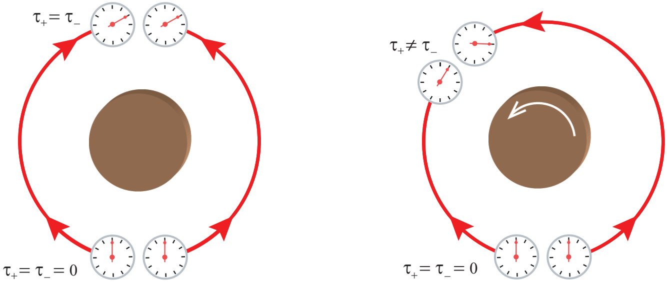
- one equaling the Sagnac time delay $4\pi\mathcal{A}_\phi$
- plus one due to the gravitomagnetic force $\gamma\vec{U}\times\vec{H}$
  - repulsive (attractive) for co-(counter) rotating geodesics
  - analogous to magnetic force produced by spinning body

For Weyl class cylinder in star fixed frame, $\vec{H}=0$

- $\Delta t_{\rm geo}$ reduces to Sagnac time delay
- What was said about beams in optical loops, applies as well to pairs of particles in circular geodesics

Gravitomagnetic clock effect

Possible to distinguish field of static from rotating cylinders using only one pair of clocks in oppositely rotating geodesics



- around a static cylinder, both clocks measure the same proper time between the events where they meet
- around a rotating cylinder, proper times differ when they meet

Local vs global staticity

Distinction between fields of static and Weyl class rotating cylinders is archetype of the contrast between *globally* static, and *locally but non-globally* static spacetimes

- Staticity: time-like killing vector field ξ^α exists such that $\xi_\alpha = \eta \partial_\alpha \psi$ (i.e., ξ_α is proportional to the gradient of a smooth function ψ)
  - *locally*: amounts to ξ^α being hypersurface orthogonal (vorticity-free)
    - satisfied by both static and rotating Weyl class cylinders
  - *globally*, vorticity-free condition not sufficient

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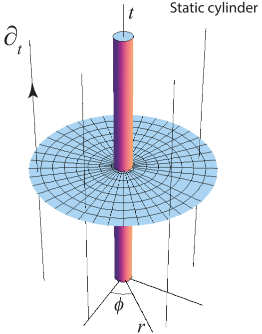
- Local staticity: a coordinate system exists such that the metric takes the stationary form $ds^2 = -e^{2\Phi}(dt - \mathcal{A}_i dx^i)^2 + h_{ij}dx^i dx^j$ with $\boldsymbol{\mathcal{A}} \equiv \mathcal{A}_i \mathrm{d}x^i$ a *closed form*, $\mathrm{d}\boldsymbol{\mathcal{A}} = 0$

- Global staticity: $\boldsymbol{\mathcal{A}}$ is moreover *exact* ($\Rightarrow \boldsymbol{\mathcal{A}} = \mathcal{A}_\phi \mathrm{d}\phi = 0$, in axistationary case)
  - field of static cylinder (Levi-Civita metric) is globally static
  - field of rotating Weyl class cylinder is locally but non-globally static

- A spacetime is locally static *iff* it admits a hypersurface orthogonal Killing vector ξ^α
- it is moreover globally static *iff* such hypersurfaces intersect each integral line of ξ^α only once (i.e., are of *global simultaneity*)



- Levi-Civita static cylinder: hypersurfaces orthogonal to ∂_t are the planes $t = const$ $\Rightarrow$ *globally static*

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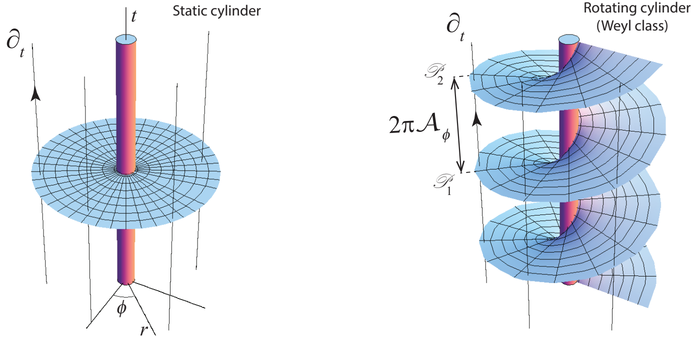

Weyl class rotating cylinder (in canonical form): hypersurfaces orthogonal to ∂_t are the *helicoids* $t - \mathcal{A}_\phi \phi = const$.

- not hypersurface of global simultaneity
(each 2π turn along ϕ lands on a different event in time; gap $= 2\pi\mathcal{A}_\phi$)
- Killing observers unable to synchronize clocks around the cylinder

Conclusion

- We have shown that the Lewis metric of the Weyl class can be put in a "canonical" form, corresponding to a system of coordinates fixed to the "distant stars"
  - depends only on 3 parameters: the Komar mass and angular momentum per unit length, plus the angle deficit
  - striking similarities with electromagnetic analogue
  - has smooth matching with Van Stockum's interior solution in star-fixed coordinates
  - allows for a transparent comparison with the Levi-Civita field of a static cylinder

- established their distinction in terms of the physical effects (gravitomagnetic effects) that detect the rotation
  - seen to differ only in the gravitomagnetic potential 1-form $\mathcal{A}$
  - manifest in the Sagnac and gravitomagnetic clock effects, and in the synchronization of clocks

- archetype of *local* vs *global* staticity: local staticity amounts to *closure* of $\mathcal{A}$, global staticity to its *exactness*