Robinson-Trautman Einstein-Maxwell fields of Petrov type D

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Einstein-Maxwell + Λ-term:

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}$$

Robinson-Trautman:

the Debever-Penrose vector $\boldsymbol{k}$ is

- geodesic and shearfree ($\kappa = \sigma = 0$)
- non-rotating ($\Im\rho = 0$)
- diverging ($\rho \neq 0$)

Aren't all these known?

(Stephani et al. : Debever 1971 "doubly aligned", Leroy 1976 all others)

- "doubly aligned": both real PND's of $\boldsymbol{F}$ are parallel to Debever-Penrose vectors
- "(half-)aligned": at least one PND of $\boldsymbol{F}$ is parallel to a Debever-Penrose vector
- "non-aligned": no PND's of $\boldsymbol{F}$ are parallel to a Debever-Penrose vector

However

- Stephani et al. is based on the standard Robinson-Trautman line-element, requiring

$$R_{ab}m^a m^b = R_{ab}k^a k^b = R_{ab}m^a k^b = 0$$

hence alignment ...

- Leroy 1976 assumes the *Maxwell* PND $\boldsymbol{k}$ to have $\kappa = \sigma = \Im\rho = 0$, implying alignment by Goldberg-Sachs ...

So the question remains

What about any non-aligned solutions?

Theorem (NVdB, Gen.Rel.Grav. 2017):

$\boldsymbol{k}$ multiple Debever-Penrose vector with $\kappa = \sigma = 0$
and
$\boldsymbol{k}$ no PND of $\boldsymbol{F} \Longrightarrow \boldsymbol{F}$ non-null and $\Lambda = 0$

Corollary for type D:

If both Debever-Penrose vectors $\boldsymbol{k}$ and $\boldsymbol{\ell}$ of a type D Einstein-Maxwell solution are geodesic and shear-free ($\kappa = \sigma = \lambda = \nu = 0$) then

$\Lambda \neq 0 \Longrightarrow$
Plebański and Demiański 7-parameter metric
(Ann.Phys. 1976)

...what if $\Lambda = 0$?

(Geroch, Held and Penrose, J.Math.Phys. 1973)

$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4 \equiv \boldsymbol{m}, \bar{\boldsymbol{m}}, \boldsymbol{\ell}, \boldsymbol{k}$ with $\boldsymbol{k} \cdot \boldsymbol{\ell} = -1, \boldsymbol{m} \cdot \bar{\boldsymbol{m}} = 1$

Under boosts

$$\boldsymbol{k} \to A\boldsymbol{k},\ \boldsymbol{\ell} \to A^{-1}\boldsymbol{\ell}$$

and spatial rotations

$$\boldsymbol{m} \to e^{i\theta}\boldsymbol{m}$$

well-weighted variables η of weight $[p, q]$ transform as

$$\eta \to A^{\frac{p+q}{2}} e^{i\frac{p-q}{2}\theta}\eta$$

(η has *boost-weight* $= \frac{p+q}{2}$ and *spin-weight* $= \frac{p-q}{2}$).

Basic variables:

$$\kappa = \Gamma_{414}, \quad \tau = \Gamma_{413}, \quad \sigma = \Gamma_{411}, \quad \rho = \Gamma_{412},$$
$$\nu = \Gamma_{233}, \quad \pi = \Gamma_{234}, \quad \lambda = \Gamma_{232}, \quad \mu = \Gamma_{231},$$

$(\Gamma_{abc} = -\Gamma_{bac} \equiv \boldsymbol{e}_a \nabla_c(\boldsymbol{e}_b))$,

$$\Phi_{00}, \Phi_{22}, \Phi_{01}, \Phi_{12}, \Phi_{02}, \Phi_{11},$$
$$R, \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4.$$

$\alpha, \beta, \epsilon, \gamma$ get absorbed in $þ, þ', ð, ð'$:

$$\begin{aligned} þ\eta &= (D - p\epsilon - q\bar{\epsilon})\eta \\ þ'\eta &= (\Delta - p\gamma - q\bar{\gamma})\eta \\ ð\eta &= (\delta - p\beta - q\bar{\alpha})\eta \\ ð'\eta &= (\bar{\delta} - p\alpha - q\bar{\beta})\eta \end{aligned}$$

Symmetry transformations:

- complex conjugation
- prime transformation:

$$\boldsymbol{k} \leftrightarrow \boldsymbol{\ell}, \boldsymbol{m} \leftrightarrow \bar{\boldsymbol{m}},$$

$$\kappa \leftrightarrow -\nu, \quad \tau \leftrightarrow -\pi, \quad \sigma \leftrightarrow -\lambda, \quad \rho \leftrightarrow -\mu,$$
$$\Phi_{ij} \leftrightarrow \Phi_{2-i\,2-j}, \quad \Psi_i \leftrightarrow \Psi_{4-i}$$

Basic equations:

- 6 complex Ricci equations

$$
\begin{aligned}
\text{Þ}\tau - \text{Þ}'\kappa &= (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Phi_{01} + \Psi_1, \\
\eth\rho - \eth'\sigma &= (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa + \Phi_{01} - \Psi_1, \\
\text{Þ}\sigma - \eth\kappa &= (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \\
\text{Þ}\rho - \eth'\kappa &= \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau' + \Phi_{00}, \\
\text{Þ}'\sigma - \eth\tau &= \sigma\rho' - \bar{\lambda}\rho - \tau^2 + \kappa\bar{\nu} - \Phi_{02}, \\
\text{Þ}'\rho - \eth'\tau &= \rho\bar{\rho}' - \lambda\sigma - \tau\bar{\tau} + \kappa\nu - \Psi_2 - \frac{1}{12}R
\end{aligned}
$$

- Maxwell equations

$$\text{Þ}\Phi_1 - \eth'\Phi_0 = \pi\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2$$
$$\text{Þ}\Phi_2 - \eth'\Phi_1 = -\lambda\Phi_0 + 2\pi\Phi_1 + \rho\Phi_2$$

- 9 complex + 2 real Bianchi equations
- commutator relations

For simplicity assume "double RT" (with doubly aligned analogues the 2-parameter Reissner and 4-parameter charged C-metric):

- $\kappa = \nu = \sigma = \lambda = 0$
- $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$
- $\rho, \mu \in \mathbb{R}$
- $\tau + \bar{\pi} = 0 \; (\boldsymbol{m} \wedge \mathrm{d}\boldsymbol{m} = 0)$
- $\Phi_0 \Phi_2 - \Phi_1^2 \neq 0$

and define extension variables

- $\mathcal{R} = þ\Phi_2, \mathcal{S} = ð\Phi_2, \mathcal{T} = -þ'\rho$

Ricci and Bianchi equations:

$$\begin{aligned}
\eth\rho &= \Phi_0\overline{\Phi_1}, \text{Þ}\rho = \rho^2 + \Phi_0\overline{\Phi_0}, \text{Þ}'\rho = -\mathcal{T}, \\
\eth\pi &= -\pi\bar{\pi} - \rho\mu + \mathcal{T} - \Psi_2, \eth'\pi = -\Phi_2\overline{\Phi_0} - \pi^2, \text{Þ}'\pi = -\Phi_2\overline{\Phi_1}, \\
\eth\Phi_1 &= \mu\Phi_0 - 2\bar{\pi}\Phi_1 - \mathcal{R}', \text{Þ}'\Phi_1 = -2\mu\Phi_1 + \bar{\pi}\Phi_2 + \mathcal{S}, \\
\eth\Phi_2 &= \mathcal{S}, \eth'\Phi_2 = 0,\ \text{Þ}\Phi_2 = \mathcal{R},\ \text{Þ}'\Phi_2 = 0, \\
\eth\Psi_2 &= 2\rho\Phi_1\overline{\Phi_2} + 2\bar{\pi}\Phi_1\overline{\Phi_1} - \overline{\Phi_2}\mathcal{S}' + \overline{\Phi_1}\mathcal{R}' - 3\bar{\pi}\Psi_2, \\
\text{Þ}\Psi_2 &= 2\rho\Phi_1\overline{\Phi_1} + 2\bar{\pi}\Phi_1\overline{\Phi_0} - \overline{\Phi_1}\mathcal{S}' + \overline{\Phi_0}\mathcal{R}' + 3\rho\Psi_2, \\
\text{Þ}'\Psi_2 &= -2\mu\Phi_1\overline{\Phi_1} + 2\pi\Phi_1\overline{\Phi_2} + \overline{\Phi_1}\mathcal{S} - \overline{\Phi_2}\mathcal{R} - 3\mu\Psi_2,
\end{aligned}$$

with $\mathcal{T}' = \mathcal{T}$ and $\overline{\Psi_2} = \Psi_2$.

$$
\begin{aligned}
\eth'\mathcal{S} &= 2(\Psi_2 - \Phi_1\overline{\Phi_1} - \rho\mu)\Phi_2,\\
\text{Þ}'\mathcal{S} &= 2(\mu\bar{\pi} - \Phi_1\overline{\Phi_2})\Phi_2 - \mu\mathcal{S},\\
\eth'\mathcal{R} &= -2(\Phi_1\overline{\Phi_0} + \pi\rho)\Phi_2 - \pi\mathcal{R},\\
\text{Þ}'\mathcal{R} &= (2\pi\bar{\pi} - 2\Phi_1\overline{\Phi_1} + \Psi_2)\Phi_2 - 3(\mu\mathcal{R} - \pi\mathcal{S}),
\end{aligned}
$$

and

$$\Phi_2\Psi_2 - \mu\mathcal{R} + \pi\mathcal{S} = 0$$

With $w = \mathcal{S}/\mu$ $(0,0)$-weighted it follows that

$$\begin{cases} \pi\eth w + \bar{\pi}\eth' w' & = \dots, \\ \mu\overline{\Phi_0}\eth w - \rho\overline{\Phi_2}\eth' w' & = \dots \end{cases}$$

- if $\mu\pi\Phi_0 + \rho\bar{\pi}\Phi_2 \neq 0$ then doubly aligned
- if $\mu\pi\Phi_0 + \rho\bar{\pi}\Phi_2 = 0$ then real $(0,0)$-weighted quantities f, g, w_0 and a constant C_0 ($|C_0| = 1$) exist such that
  - $\Phi_0 = \rho\bar{\pi}C_0 f,\ \Phi_1 = C_0 g,\ \Phi_2 = -\mu\pi C_0 f$,
  - $w = C_0 w_0$
  - $\mathcal{T} = \Psi_2 + \rho\mu(1 + f^2\pi\bar{\pi}) - w_0 f^{-1}$

- fix the tetrad such that

$$\mu = e\rho \; (e = \pm 1) \text{ and } \pi = -\tau \, (\in \mathbb{R}),$$

hence

$$\Phi_0 = \Phi_2$$

- define

$$h = w_0 - 2g, \; j = f\Psi_2 + 2g - w_0,$$
$$\boldsymbol{\Omega}^1 = e\omega^3 - \omega^4, \; \boldsymbol{\Omega}^2 = \omega^1 + \omega^2$$

Cartan's equations become

$$\begin{aligned} \mathrm{d}\boldsymbol{\Omega}^1 &= -\pi\boldsymbol{\Omega}^1 \wedge \boldsymbol{\Omega}^2, \; \mathrm{d}\boldsymbol{\Omega}^2 = \rho\boldsymbol{\Omega}^1 \wedge \boldsymbol{\Omega}^2, \\ \mathrm{d}\omega^3 &= \omega^3 \wedge (-\pi\omega^1 - \pi\omega^2 + e\tfrac{2g-j}{2\rho f}\omega^4), \\ \mathrm{d}\omega^4 &= \omega^4 \wedge (-\pi\omega^1 - \pi\omega^2 - \tfrac{2g-j}{2\rho f}\omega^3) \end{aligned}$$

while

$$
\begin{aligned}
\mathrm{d}f &= f[(f^2\pi^2\rho - fg\rho + \rho)\boldsymbol{\Omega}^1 + (-ef^2\pi\rho^2 - fg\pi + \pi)\boldsymbol{\Omega}^2],\\
\mathrm{d}g &= -\boldsymbol{\Omega}^1(f\pi^2 - h)\rho + \pi(ef\rho^2 - j)\boldsymbol{\Omega}^2,\\
\mathrm{d}h &= h[(f^2\pi^2\rho - 2\rho)\boldsymbol{\Omega}^1 - (fg+2)\pi\boldsymbol{\Omega}^2],\\
\mathrm{d}j &= j[-(fg+2)\rho\boldsymbol{\Omega}^1 + (-ef^2\pi\rho^2 - 2\pi)\boldsymbol{\Omega}^2],\\
\mathrm{d}\rho &= -\frac{1}{2ef}(2ef^3\pi^2\rho^2 + 2ef\rho^2 - 2g + j)\boldsymbol{\Omega}^1 + \boldsymbol{\Omega}^2 fg\pi\rho,\\
\mathrm{d}\pi &= \frac{1}{2f}(2ef^3\pi^2\rho^2 - 2f\pi^2 - 2g - h)\boldsymbol{\Omega}^2 + \boldsymbol{\Omega}^1 fg\pi\rho.
\end{aligned}
$$

- integrable!
- abelian G_2

Introduce coordinates such that

$$\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2 = i\mathcal{P}\mathrm{d}z,\;\; e\boldsymbol{\omega}^3 + \boldsymbol{\omega}^4 = \mathcal{Q}\mathrm{d}t,$$
$$\boldsymbol{\Omega}^1 = \mathcal{B}\mathrm{d}u,\;\; \boldsymbol{\Omega}^2 = \mathcal{C}\mathrm{d}v,$$

Then $\pi = \mathcal{B}_{,u}/(\mathcal{B}\mathcal{C})$ and $\rho = \mathcal{C}_{,v}/(\mathcal{B}\mathcal{C})$ with $\mathcal{B}/\mathcal{C}$ separable in u, v and hence $\mathcal{B} = \mathcal{C}$, while $\mathcal{P}$ and $\mathcal{Q}$ follow from

$$\begin{aligned} \mathrm{d}\log(\tfrac{\mathcal{Q}}{\mathcal{B}\rho}) &= \mathcal{B}f\pi(\rho\pi f\mathrm{d}v - g\mathrm{d}u), \\ \mathrm{d}\log(\tfrac{\mathcal{P}}{\mathcal{B}\pi}) &= -\mathcal{B}f\rho(e\rho\pi f\mathrm{d}u + g\mathrm{d}v). \end{aligned}$$

When $hj \neq 0$ functions $\varsigma = \varsigma(u), \xi = \xi(v)$ exist such that

$$\begin{aligned} \mathcal{P} &= \mathcal{B}\varsigma, \;\; \mathcal{Q} = \mathcal{B}\xi, \\ \pi &= \frac{\varsigma}{j\mathcal{B}^2}, \;\; \rho = \frac{\xi}{h\mathcal{B}^2}, \end{aligned}$$

and

$$\begin{aligned} f &= hj\mathcal{B}^5, \\ g &= ej\mathcal{B}^2\xi' + e\frac{j\mathcal{B}\xi^2}{h} + \tfrac{1}{2}j \\ &= -h\mathcal{B}^2\varsigma' - \frac{h\mathcal{B}\varsigma^2}{j} - \tfrac{1}{2}h, \end{aligned}$$

with the metric given by,

$$\mathrm{d}s^2 = \frac{\mathcal{B}^2}{2}(\mathrm{d}u^2 + e\mathrm{d}v^2 - e\xi^2\mathrm{d}t^2 + \varsigma^2\mathrm{d}z^2),$$

but . . . with a nasty system of pde's (for $h, j, \mathcal{B}$) and ode's (for ς, ξ).

Defining $y = \int \varsigma \mathrm{d}u, x = \int \xi \mathrm{d}v$

$$\mathrm{d}s^2 = \frac{k^2\mathcal{B}^2}{2}(\varsigma^{-2}\mathrm{d}y^2 + e\xi^{-2}\mathrm{d}x^2 - e\xi^2\mathrm{d}t^2 + \varsigma^2\mathrm{d}z^2),$$

with

$$\begin{aligned}
\mathcal{B}^{-2} &= xy^2 - yx^2 - \frac{p}{2}(x-y)^2 - 2e(2x-y)\Xi_0 + 2(x-2y)\Sigma_0 + q \\
\xi^2 &= \tfrac{e}{4}x^4 + 3\Xi_0 x^2 - (p(e\Sigma_0 + \Xi_0) + eq)x + e(2\Sigma_0 - \Xi_0)^2 + \tfrac{1}{2}epq \\
\varsigma^2 &= -\tfrac{1}{4}y^4 + 3\Sigma_0 y^2 + (p(e\Xi_0 + \Sigma_0) - q)y - (2\Xi_0 - e\Sigma_0)^2 - \tfrac{1}{2}pq
\end{aligned}$$

When $h = 0 \neq j$,

$$\begin{aligned}
\mathcal{B}^{-2} &= (x - 6e\frac{\Sigma_0}{\Xi_0})y^2 + 3\frac{eq}{\Xi_0}y - \frac{e\Xi_0}{3}x^2 - 2\Sigma_0 x + e\frac{\Xi_0 p - 12\Sigma_0^2}{\Xi_0} \\
\xi^2 &= \frac{\Xi_0}{3}x^3 - px + \frac{3e}{4\Xi_0^2}(8\Xi_0\Sigma_0 p - 96\Sigma_0^3 + 3q^2) \\
\varsigma^2 &= -\tfrac{1}{4}y^4 + 3\Sigma_0 y^2 - \tfrac{p}{3}\Xi_0 - qy + 3\Sigma_0^2
\end{aligned}$$

When $j = 0 \neq h$,

$$\begin{aligned}
\mathcal{B}^{-2} &= (6e\frac{\Xi_0}{\Sigma 0} - y)x^2 - \frac{\Sigma_0}{3}y^2 + 3\frac{eq}{\Sigma_0}x - 2e\Xi_0 y + \frac{e\Sigma_0 p - 12\Xi_0^2}{\Sigma_0} \\
\xi^2 &= \tfrac{e}{4}x^4 + 3\Xi_0 x^2 + \tfrac{p}{3}\Sigma_0 + qx - 3e\Xi_0^2 \\
\varsigma^2 &= \frac{\Sigma_0}{3}y^3 - epy + \frac{3}{4\Sigma_0^2}(8\Xi_0\Sigma_0 p - 96e\Xi_0^3 - 3q^2)
\end{aligned}$$

- 3 distinct 5-parameter solutions
- abelian G_2 and admit a valence 2 Killing spinor
- all solutions with $e = +1$ are
  - static in the domain where $\mathcal{B}^2, \varsigma^2, \xi^2$ are positive, with time-like Killing vector ∂_t,
  - have parallel electrostatic and magnetostatic field vectors (cf. Das J Math Phys 1979)

- re-defining the parameters and transforming $(t, z, y, t) \to$

$$(\sqrt{2}A^{-1}ta^{-2}, \sqrt{2}A^{-1}za^{-2}, (mAy + \tfrac{1}{6})a^4, (mAx - \tfrac{1}{6})a^4),$$

  the $hj \neq 0$ solution yields for $a \to 0$ the vacuum C-metric,

$$\mathrm{d}s^2 = \frac{1}{A^2(x+y)^2}(-F\mathrm{d}t^2 + G\mathrm{d}z^2 + \frac{1}{F}\mathrm{d}x^2 + \frac{1}{G}\mathrm{d}y^2), \quad (1)$$

  $(F(x) = -1 + x^2 - 2Amx^3$ and $G(y) = -F(-x))$.
- the charged C-metric $(F(x) = -1 + x^2 - 2Amx^3 + q^2x^4$, $G(y) = -F(-y))$ appears not to be a limit

Thank you!

Non-aligned E-M