Dark-fluid constraints of shear-free universes

Amare Abebe

Amare Abebe
Center for Space Research
North-West University, Mahikeng, South Africa

Spanish-Portuguese Relativity Meeting
13 - 16 September 2021, Aveiro, Portugal

September 16, 2021

Introduction

- The recent discovery of the accelerated rate of cosmic expansion has inspired a wave of new research into the nature of gravitational physics
  - ✓ New alternatives and/or generalisations to Einstein's General Relativity (GR) theory abound already
- In order to understand the dynamics of nonlinear fluid flows, it is important to understand the relationship between their Newtonian and general relativistic limits
  - ✓ Relevant both in the physics of gravitational collapse and the late (nonlinear) stages of cosmic structure formation
- The differential properties of time-like geodesics describe the fluid flows in cosmology
  - ✓ The expansion Θ, shear (distortion) $\sigma_{\alpha\beta}$, rotation (vorticity) ω^{α}, and acceleration A_a of the four-velocity field u^a tangent to the fluid flowlines describe kinematics of such fluid flows
- Important to make a consistency analysis of the field equations for different models where integrability conditions arise from imposing external restrictions
- The introduction of integrability conditions in CG-dominated universes helps us explore the existence and nature of these universe models that would otherwise not exist under the standard matter (dust) conditions
- Here we explore general properties of classes of shear-free spacetimes characterised by the vanishing of shear $\sigma_{\alpha\beta}$, but generally non-vanishing energy density μ , vorticity ω_{α} and a locally free gravitational field covariantly described by the gravito-electric (GE) and gravito-magnetic (GM) components of the Weyl tensor, $E_{\alpha\beta}$ and $H_{\alpha\beta}$, respectively

- The standard GR gravitational action with a matter field contribution to the Lagrangian, $\mathcal{L}_m$, is given by[1]

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g}\,[R + 2\mathcal{L}_m]$$

- Using the variational principle of least action with respect to the metric g_{ab}, the generalised Einstein Field Equations (EFEs) can be given in a compact form as

$$G_{\alpha\beta} = T_{\alpha\beta}$$

  with the first (geometric) term represented by the Einstein tensor, and energy-momentum tensor of matter fluid forms given by

$$T_{\alpha\beta} = \mu u_\alpha u_\beta + p h_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha + \pi_{\alpha\beta}$$

  - ✓ μ, p, q_a and $\pi_{\alpha\beta}$ are the energy density, isotropic pressure, heat flux and anisotropic pressure of the fluid, respectively
  - ✓ $u^\alpha \equiv \frac{dx^\alpha}{dt}$ is the 4-velocity of fundamental observers comoving with the fluid and is used to define the *covariant time derivative* for any tensor $S^{\alpha..\beta}{}_{\gamma..\delta}$ along an observer's worldlines:
$$\dot{S}^{\alpha..\beta}{}_{\gamma..\delta} = u^\lambda \nabla_\lambda S^{\alpha..\beta}{}_{\gamma..\delta}$$

[1] We have used $8\pi G = 1 = c$

Covariant description...

- The projection tensor into the tangent 3-spaces orthogonal to u^α is given by $h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta$ and is used to define the fully orthogonally *projected covariant derivative* for any tensor $S^{\alpha..\beta}{}_{\gamma..\delta}$:

$$\tilde{\nabla}_\lambda S^{\alpha..\beta}{}_{\gamma..\delta} = h^\alpha{}_\mu h^\nu{}_\gamma ... h^\beta{}_\theta h^\phi{}_\delta h^\tau{}_\lambda \nabla_\tau S^{\mu..\theta}{}_{\nu..\phi}$$

  with total projection on all the free indices

- The orthogonally *projected symmetric trace-free* (PSTF) part of vectors and rank-2 tensors is defined as

$$V^{\langle\alpha\rangle} = h^\alpha{}_\beta V^\beta \qquad S^{\langle\alpha\beta\rangle} = \left[h^{(\alpha}{}_\gamma h^{\beta)}{}_\delta - \tfrac{1}{3} h^{\alpha\beta} h_{\gamma\delta}\right] S^{\gamma\delta}$$

  and the volume element for the rest spaces orthogonal to u^α is given by

$$\epsilon_{\alpha\beta\gamma} = u^\delta \eta_{\delta\alpha\beta\gamma} = -\sqrt{|g|}\delta^0{}_{[\alpha}\,\delta^1{}_\beta \delta^2{}_\gamma \delta^3{}_{\delta]} u^\delta$$

  where η_{abcd} is the 4-dimensional volume element with the properties

$$\eta_{\alpha\beta\gamma\delta} = \eta_{[\alpha\beta\gamma\delta]} = 2\epsilon_{\alpha\beta[\gamma} u_{\delta]} - 2u_{[\alpha}\epsilon_{\beta]\gamma\delta}$$

- Covariant spatial divergence and curl of vectors and rank-2 tensors:

$$\mathrm{div}\, V = \tilde{\nabla}^\alpha V_\alpha \qquad (\mathrm{div}\, S)_\alpha = \tilde{\nabla}^\beta S_{\alpha\beta}$$

$$curl V_\alpha = \epsilon_{\alpha\beta\gamma} \tilde{\nabla}^\beta V^\gamma \qquad curl S_{\alpha\beta} = \epsilon_{\gamma\delta(\alpha} \tilde{\nabla}^\gamma S_{\beta)}{}^\delta$$

- The first covariant derivative of u^α can be split into its irreducible parts as

$$\nabla_\alpha u_\beta = -A_\alpha u_\beta + \tfrac{1}{3}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \epsilon_{\alpha\beta\gamma}\omega^\gamma$$
$$A_\alpha \equiv \dot{u}_\alpha \quad \Theta \equiv \tilde{\nabla}_\alpha u^\alpha \quad \sigma_{\alpha\beta} \equiv \tilde{\nabla}_{\langle\alpha} u_{\beta\rangle} \quad \omega^\alpha \equiv \epsilon^{\alpha\beta\gamma}\tilde{\nabla}_\beta u_\gamma$$

- The *Weyl conformal curvature tensor* $C_{\alpha\beta\gamma\delta}$ is defined as

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2g^{[\alpha}{}_{[\gamma}R^{\beta]}{}_{\delta]} + \frac{R}{3}g^{[\alpha}{}_{[\gamma}g^{\beta]}{}_{\delta]}$$

  and can be split into its "electric" and "magnetic" parts, respectively, as

$$E_{\alpha\beta} \equiv C_{\alpha\gamma\beta\delta}u^\gamma u^\delta$$
$$H_{\alpha\beta} \equiv \tfrac{1}{2}\eta_{\alpha\theta}{}^{\gamma\delta}C_{\gamma\delta\beta\lambda}u^\theta u^\lambda$$

  - ✓ $E_{\alpha\beta}$ represents the free gravitational field (tidal forces)
  - ✓ $H_{\alpha\beta}$ is responsible for gravitational waves, no Newtonian analogue

- Cosmological quantities that vanish in the background spacetime are considered to be first-order and gauge-invariant by virtue of the Stewart-Walker lemma

- The Chaplygin gas is a dark-fluid model whose EoS

$$p = -\frac{A}{\mu} \qquad \text{A=const}$$

  allows for a solution of the form:

$$\mu(a) = \sqrt{A + \frac{B}{a^6}}$$

  - ✓ Early universe: $\mu \sim a^{-3}$, behaves as dust (dark matter and baryonic matter)
  - ✓ Late universe: $\mu \sim \sqrt{A}$, behaves like dark energy
- Widely explored model for a unified description of the cosmological background expansion history
- Not so-widely explored at the perturbations level
- Consistency relations that might constrain the model?

Linearized field equations

- In a multi-component fluid universe filled with standard matter fields (dust, radiation, etc) and Chaplygin-gas contributions, the total energy density, isotropic and anisotropic pressures and heat flux are given, respectively, by

$$\mu \equiv \sum_i \mu^i \qquad p \equiv \sum_i p^i \qquad q_\alpha \equiv \sum_i q^i_\alpha \qquad \pi_{\alpha\beta} \equiv \sum_i \pi^i_{\alpha\beta}$$

  with the index i labelling the thermodynamic property of the *ith* fluid

- If we assume the late time matter distribution to be dominated by dust and the CG, then we can write:

$$\mu = \mu^d + \mu^c \qquad p = p^c \qquad q_\alpha = q^d_\alpha + q^c_\alpha$$

  where dust is taken to be pressureless and we have further assumed that the anisotropic pressures identically vanish at linear order: $p^d = 0$, $\pi^d_{\alpha\beta} = 0 = \pi^c_{\alpha\beta}$

- Moreover, we assume the normalized 4-velocity u_α of fundamental observers coincides with that of standard matter u^d_α such that

$$v^c_\alpha \equiv u^c_\alpha - u_\alpha \qquad q^d_\alpha = 0 \qquad q^c_\alpha = (\mu^c + p^c)v^c_\alpha$$

  where the normalized 4-velcoity of the Chaplygin fluid is tilted w.r.t to u_α by the peculiar velocity $v^c_\alpha \ll 1$ (a non-relativistic approximation)

Linearized field equations...

- The complete linearised propagation and constraint equations are given by:

$$\dot{\mu}^d = -\mu^d \Theta \tag{1.1}$$
$$\dot{\mu}^c = -(\mu^c + p^c)\Theta - \tilde{\nabla}^\alpha q^c_\alpha \tag{1.2}$$
$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p^c) + \tilde{\nabla}_\alpha A^\alpha \tag{1.3}$$
$$\dot{q}^c_\alpha = -\frac{4}{3}\Theta q^c_\alpha - (\mu^c + p^c)A_\alpha - \tilde{\nabla}_\alpha p^c \tag{1.4}$$
$$\dot{\omega}_\alpha = -\frac{2}{3}\Theta\omega_\alpha - \frac{1}{2}\epsilon_{\alpha\beta\gamma}\tilde{\nabla}^\beta A^\gamma \tag{1.5}$$
$$\dot{\sigma}_{\alpha\beta} = -\frac{2}{3}\Theta\sigma_{\alpha\beta} - E_{\alpha\beta} + \tilde{\nabla}_{\langle\alpha}A_{\beta\rangle} \tag{1.6}$$
$$\dot{E}_{\alpha\beta} = \epsilon_{\gamma\delta\langle\alpha}\tilde{\nabla}^\gamma H^\delta{}_{\beta\rangle} - \Theta E_{\alpha\beta} - \frac{1}{2}(\mu + p)\,\sigma_{\alpha\beta} - \frac{1}{2}\tilde{\nabla}_{\langle\alpha}q^c_{\beta\rangle} \tag{1.7}$$
$$\dot{H}_{\alpha\beta} = -\Theta H_{\alpha\beta} - \epsilon_{\gamma\delta\langle\alpha}\tilde{\nabla}^\gamma E^\delta{}_{\beta\rangle} \tag{1.8}$$

$$(C^1)_\alpha := \tilde{\nabla}^\beta\sigma_{\alpha\beta} - \frac{2}{3}\tilde{\nabla}_\alpha\Theta + \epsilon_{\alpha\beta\gamma}\tilde{\nabla}^\beta\omega^\gamma + q^c_\alpha = 0 \tag{1.9}$$
$$(C^2)_{\alpha\beta} := \epsilon_{\gamma\delta(\alpha}\tilde{\nabla}^\gamma\sigma^\delta{}_{\beta)} + \tilde{\nabla}_{\langle\alpha}\omega_{\beta\rangle} - H_{\alpha\beta} = 0 \tag{1.10}$$
$$(C^3)_\alpha := \tilde{\nabla}^\beta H_{\alpha\beta} + (\mu + p^c)\omega_\alpha + \frac{1}{2}\epsilon_{\alpha\beta\delta}\tilde{\nabla}^\beta q^\delta_c = 0 \tag{1.11}$$
$$(C^4)_\alpha := \tilde{\nabla}^b E_{\alpha\beta} - \frac{1}{3}\tilde{\nabla}_\alpha\mu + \frac{1}{3}\Theta q^c_\alpha = 0 \tag{1.12}$$
$$(C^5) := \tilde{\nabla}^\alpha\omega_\alpha = 0 \tag{1.13}$$

- Not silent models: spatial derivatives coupled with the evolution equations
  - ✓ Flowlines on any hypersurface do not evolve separately from each other

- Over the years, the role of shear in GR and the special nature of shear-free cases in particular have been studied
- Gödel showed [2] that shear-free time-like geodesics of some spatially homogeneous universes cannot expand and rotate simultaneously and this result was later generalized[34] to include inhomogeneous cases of shear-free time-like geodesics
- An interesting aspect of the shear-free condition is that it does not hold in Newtonian gravitation theory although Newtonian theory is a limiting case of GR under special circumstances, namely at low-speed relative motion of matter with no gravito-magnetic effects (vanishing magnetic part of the Weyl tensor) and hence no gravitational waves
- Let us now investigate the effect of switching off the shear term from the above evolution and constraint equations
  - ✓ A first observation is that Eq. (1.6) turns into a new constraint equation:

$$(C^6)_{\alpha\beta} := E_{\alpha\beta} - \tilde{\nabla}_{\langle\alpha} A_{\beta\rangle} = 0 \tag{2.1}$$

[2] Gödel K. Rotating universes in general relativity theory. In Proceedings of the International Congress of Mathematicians, Cambridge, Mass. 1952, Vol. 1, 175 (1952)

[3] Ellis, G. Dynamics of pressure-free matter in general relativity. Journal of Mathematical Physics 8, 1171 (1967)

[4] Nzioki, A. M., Goswami, R., Dunsby, P. K. & Ellis, G. F. Shear-free perturbations of Friedmann-Lema$\hat{?}$ire-Robertson-Walker universes

- The special case where $q^c_\alpha = 0$ where Eq. (1.4) turns into a further constraint

$$A_\alpha = -\frac{\tilde{\nabla}_\alpha p^c}{\mu^c + p^c} \tag{2.2}$$

was recently investigated [5] and shown to have counter-examples to the generalized Ellis shear-free conjecture

  - ✓ Simultaneously expanding ($\Theta \neq 0$) and rotating ($\omega^a \neq 0$) fluid flow solutions exist in Chaplygin-gas dominated cosmological models
  - ✓ These counter-examples force a special algebraic relationship between the defining CG fluid parameters
  - ✓ Beyond these counter-examples, any expanding shear-free CG-dominated universe with vanishing heat flux must generally be non-rotating

- A direct implication of this will be that from Eq. (1.5) another new constraint emerges:

$$\epsilon_{\alpha\beta\gamma}\tilde{\nabla}^\beta A^\gamma = 0 \implies A_\alpha = \tilde{\nabla}_\alpha \phi \tag{2.3}$$

i.e., if the curl of the acceleration vector A_α is zero, then A_α an be written as the grant of some scalar potential ϕ. Comparing Eqs. (2.2) and (2.3), one concludes

$$\phi = -\frac{1}{2}\ln\left(\frac{\mu^c + p^c}{\mu^c}\right) = -\frac{1}{2}\ln\left(1 - \frac{A}{\mu_c^2}\right) \tag{2.4}$$

[5] Abebe, A., Al Ajmi, M., Elmardi, M., Nandan, H. & Sabah, N. Shear-free conditions of a Chaplygin-gas-dominated universe. Int. J. Geom. Methods Mod. Phys. 2150192 (2021)

- An even more interesting consequence of the above special case (shear-free, vanishing heat flux) will be that, by virtue of the constraint equation (1.10), the gravito-magnetic component of the Weyl tensor identically vanishes, leading to the so-called *quasi-Newtonian* universe with a homogeneous expansion, since $\tilde{\nabla}_\alpha \Theta = 0$ in Eq. (1.9)
  - ✓ Such models are generally unstable to linear perturbations and do not support large-scale structure formation because

$$\tilde{\nabla}_\alpha \Theta = 0 \implies \tilde{\nabla}_\alpha \mu = 0 \implies \tilde{\nabla}_\alpha \mu^d + \tilde{\nabla}_\alpha \mu^c = 0 \tag{2.5}$$

  - ✓ This shows that there has to be a fine balance between dust and the CG such that any tendency for structures to grow out of dust perturbations will be discouraged by those of the latter
- Let us now consider shear-free models with a net heat-flux due to the CG fluid
- Limit focus to irrotational-fluid cases for now. An immediate consequence of the irrotational-fluid assumption would be that from Eq. (1.11),

$$\epsilon_{\alpha\beta\delta} \tilde{\nabla}^\beta q_c^\delta = 0 \implies q_\alpha^c = \tilde{\nabla}_\alpha \psi \tag{2.6}$$

for some scalar potential ψ

- Eq. (1.9) suggests that

$$q^c_\alpha = \frac{2}{3}\tilde{\nabla}_\alpha \Theta \implies \psi = \frac{2}{3}\Theta + C \tag{2.7}$$

for some (spatial) constant C

- We can then show that the peculiar velocity of the CG fluid (w.r.t the worldline of the fundamental observers) can be given, either in terms of the expansion gradient or total energy density gradient, by

$$v^c_\alpha = \frac{2}{3(\mu^c + p^c)}\tilde{\nabla}_\alpha \Theta = \frac{1}{(\mu^c + p^c)\Theta}\tilde{\nabla}_\alpha \mu \tag{2.8}$$

- And finally, using this result together with Eq. (1.4) the acceleration of the fluid can be shown to be

$$A_\alpha = \dot{v}^c_\alpha + \left(\frac{1}{3} + \frac{A}{\mu_c^2}\right)\Theta v^c_\alpha + \frac{A}{\mu_c^2(\mu^c + p^c)}\tilde{\nabla}_\alpha \mu^c \tag{2.9}$$

an interesting result that generalizes the quasi-Newtonian relation obtained for pure dust [6]

[6] Maartens, R. Covariant velocity and density perturbations in quasi-Newtonian cosmologies. Physical Review D 58, 124006 (1998)

- These are a class of purely gravito-magnetic irrotational models with

$$E_{\alpha\beta} = 0 \tag{2.10}$$

  and are considered to be the farthest from the Newtonian theory
- Drawing parallels to the previous subsection in which both the shear and the heat flux were switched off, one immediate observes that such spacetimes cannot be shear-free, for if we allow the shear to vanish:
  - ✓ $H_{\alpha\beta}$ would have to vanish as well
  - ✓ A vanishing heat flux means

$$\tilde{\nabla}_\alpha \Theta = 0 = \tilde{\nabla}_\alpha \mu \implies \text{FLRW background spacetime} \tag{2.11}$$

- If the heat flux does not vanish, we obtain the same results for the forms of q^c_α , v^c_α and A_α with the extra condition that

$$\tilde{\nabla}_{\langle\alpha} q^c_{\beta\rangle} = 0 \implies \tilde{\nabla}_{\langle\alpha} v^c_{\beta\rangle} = 0 = \tilde{\nabla}_{\langle\alpha} \tilde{\nabla}_{\beta\rangle} \mu \tag{2.12}$$

  with the last two equalities holding true to linear order in the perturbations
- In the purely gravito-magnetic sense of the anti-Newtonian models, the vanishing $E_{\alpha\beta}$ assumption with cosmic shear results in propagation equation (1.7) turning into a new constraint:

$$(C^7)_{\alpha\beta} = \epsilon_{\gamma\delta\langle\alpha} \tilde{\nabla}^\gamma H^\delta{}_{\beta\rangle} - \frac{1}{2}(\mu + p)\,\sigma_{\alpha\beta} - \frac{1}{2}\tilde{\nabla}_{\langle\alpha} q^c_{\beta\rangle} \tag{2.13}$$

- The constraint equation (1.12) leads to the relation

$$\tilde{\nabla}_\alpha \mu = \Theta q^c_\alpha \tag{2.14}$$

  which, together with Eq. (1.11) results in the important result

$$\tilde{\nabla}^\beta H_{\alpha\beta} = 0 \tag{2.15}$$

- Eq. (2.14) shows that q^c_α can be written as the gradient of a scalar, and the curl of the gradient of a scalar is zero for irrotational cases, then the last term in Eq. (1.11) vanishes
- A necessary condition for the propagation of gravitational radiation is the vanishing of the divergence of a non-vanishing $H_{\alpha\beta}$
  - ✓ Eq. (2.15) therefore shows that gravitational radiation can propagate in a CG-dominated anti-Newtonian universe. This is the antithesis of a Newtonian solution where gravitational wave propagation is not allowed
- Referring back to Eq. (1.9), we see that the heat flux for the anti-Newtonian model is given by

$$q^c_\alpha = \frac{2}{3}\tilde{\nabla}_\alpha \Theta - \tilde{\nabla}^\beta \sigma_{\alpha\beta} \tag{2.16}$$

- Comparing this with Eq. (2.14) and using the Friedman constraint $\Theta^2 = 3\mu$ gives rise to a new constraint on the shear:

$$\tilde{\nabla}^\beta \sigma_{\alpha\beta} = 0 \tag{2.17}$$

  - ✓ The solutions for q^c_α , v^c_α and A_α that we found in Eqs. (2.7), (2.8) and (2.9) retain their forms in this subclass of models as well, subject to the constraint (2.13)

- The CG model as a possible dark fluid alternative
- Shear-free spacetimes
  - ✓ Quasi-Newtonian solution
  - ✓ Anti-Newtonian solutions
- Possible new frontiers:
  - ✓ Analysis of the density and velocity perturbations
  - ✓ Nonlinear generalizations
  - ✓ More generalized CG models
  - ✓ Observational constraints