Compact elastic objects in general relativity

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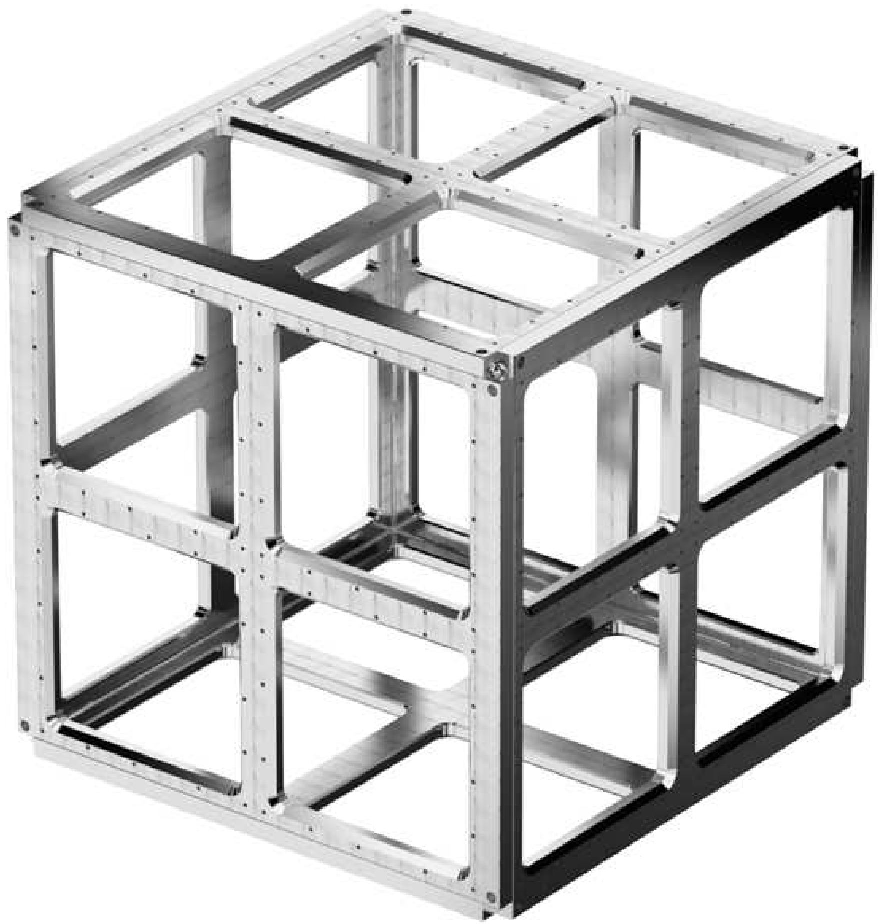
Outline

- Rigid bodies and Ehrenfest's paradox
- Relativistic elasticity
- Example: rigid rod
- Spherical symmetry
- Material models
- Numerical results
- Conclusion and outlook

Rigid bodies and Ehrenfest's paradox

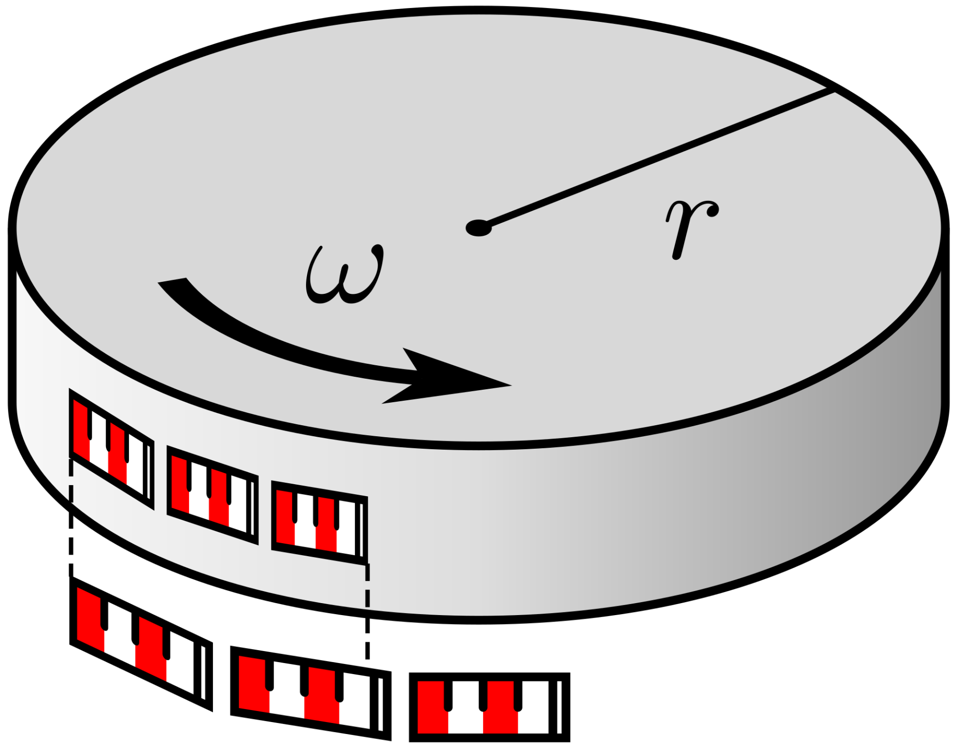


- Rigid body: distance between any two points at a given instant remains constant.

- So no rigid bodies in relativity.

- Physically, it takes time for one end of the body to receive information about forces acting on the other end.

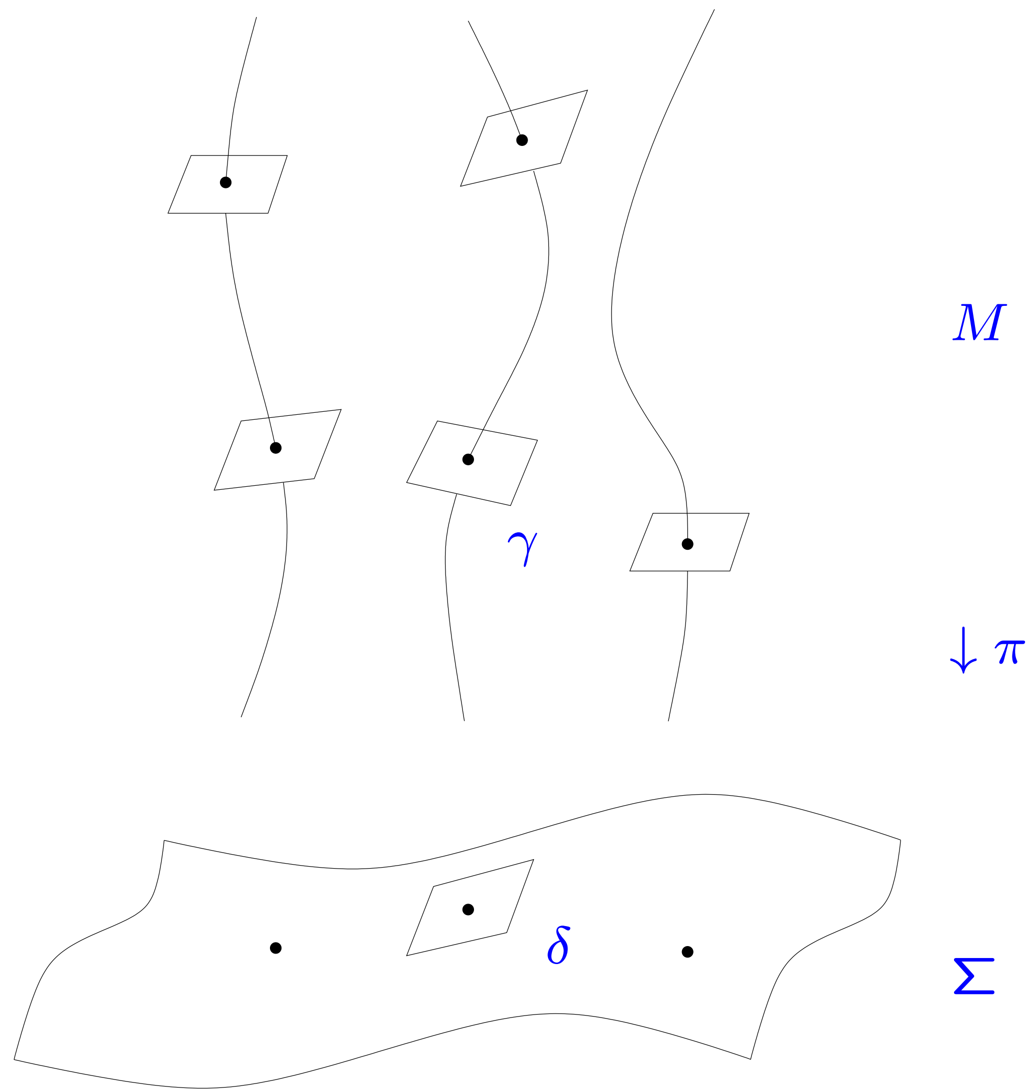
- Ehrenfest's paradox:



- So no undeformable bodies in relativity.

Relativistic elasticity

- An elastic medium in general relativity is described by:
  - A spacetime (M, g);
  - A Riemannian 3-manifold (Σ, δ) (reference configuration);
  - A projection map $\pi : M \to \Sigma$ whose level sets are timelike curves (the worldlines of the medium particles).


M
γ
$\downarrow \pi$
δ
Σ

- If we choose local coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ on Σ then we can think of π as a set of three scalar fields.

- We can complete $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ into coordinates $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ for (M, g) yielding the rest frame of any given worldline:

$$g = -d\bar{t}^2 + \gamma_{ij} d\bar{x}^i d\bar{x}^j \qquad \text{(at that worldline).}$$

- Note that

$$\gamma = \gamma_{ij} d\bar{x}^i d\bar{x}^j$$

is a (time-dependent) Riemannian metric on Σ, describing the local deformations of the medium along each worldline.

- We can compute the (inverse) metric γ from

$$\gamma^{ij} = g^{\mu\nu} \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial \bar{x}^j}{\partial x^\nu}.$$

- Choose a Lagrangian density $\mathcal{L}$ of the form $\mathcal{L} = \mathcal{L}(\bar{x}^i, \delta_{ij}, \gamma^{ij})$ for the action

$$S = \int_M \mathcal{L} \sqrt{-g}\, d^4x.$$

The energy-momentum tensor is then

$$T_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}\, g_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial \gamma^{ij}}\, \partial_\mu \bar{x}^i\, \partial_\nu \bar{x}^j - \mathcal{L}\, g_{\mu\nu}.$$

- Therefore

$$\mathcal{L} = T_{\bar{0}\bar{0}} = \rho$$

  is the rest energy density.

- The choice of $\rho = \rho(\bar{x}^i, \delta_{ij}, \gamma^{ij})$ is called the elastic law.

- Perfect fluid: $\rho = \rho(n) \Rightarrow p = n\dfrac{d\rho}{dn} - \rho$, where $n = \sqrt{\mathrm{det}(\gamma^{ij})}$ (assumes that δ_{ij} is the Kronecker delta).

Example: rigid rod

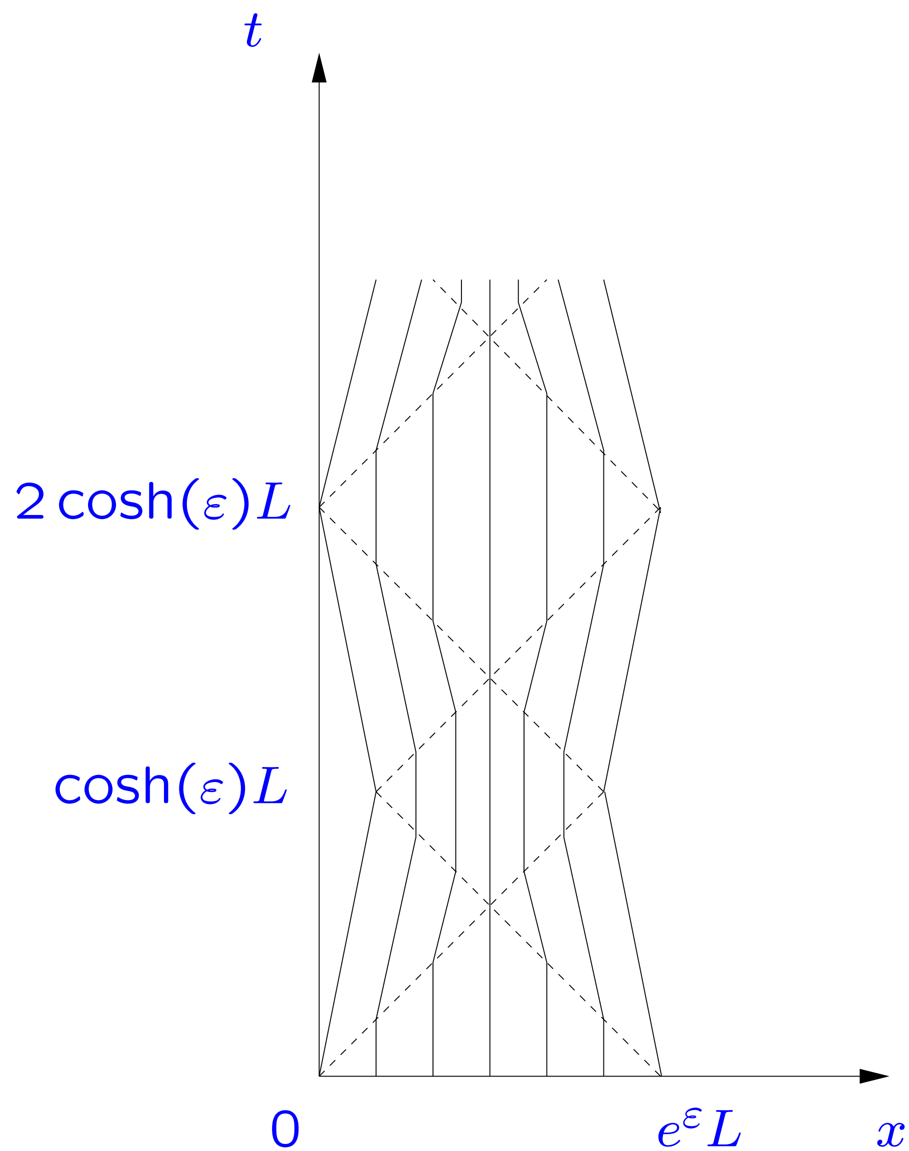
- For one-dimensional elastic bodies in a two-dimensional space-time (M, g) there is no difference between solids and fluids: the Lagrangian depends only on $\gamma^{11} = \partial_\alpha \bar{x} \partial^\alpha \bar{x}$.

- For a rigid elastic body (speed of sound = speed of light) we have $\rho = p + \rho_0$, yielding

$$\rho = \frac{\rho_0}{2}(\gamma^{11} + 1) = \frac{\rho_0}{2}(\partial_\alpha \bar{x} \partial^\alpha \bar{x} + 1).$$

- This is the Lagrangian for a massless scalar field, and so the equation of motion is just the wave equation $\Box\bar{x} = 0$, which can be exactly solved in two dimensions.

- For example, the motion of an uniformly stretched rigid rod released from rest (imposing zero pressure at the endpoints) in Minkowski's two-dimensional spacetime is as follows:



t
$2\cosh(\varepsilon)L$
$\cosh(\varepsilon)L$
0
$e^{\varepsilon}L$
x

Spherical symmetry

- Metric: $g = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$.

- TOV equations:

$$\begin{cases} \dfrac{dp_{\text{rad}}}{dr} = \dfrac{2}{r}(p_{\text{tan}} - p_{\text{rad}}) - (p_{\text{rad}} + \rho)\dfrac{d\alpha}{dr} \\ \\ \dfrac{d\alpha}{dr} = \dfrac{e^{2\beta}}{r}\left(\dfrac{m}{r} + 4\pi r^2 p_{\text{rad}}\right) \end{cases},$$

where

$$e^{-2\beta(r)} = 1 - \frac{2m(r)}{r}, \qquad m(r) = 4\pi\int_0^r \rho(u)u^2du.$$

- Equation of state: $\rho(r) = \widehat{\rho}(\delta(r), \eta(r))$, where

$$\begin{cases} \delta(r) = \sqrt{\det(\gamma^{ij})} \\ \eta(r) = \dfrac{3}{r^3} \displaystyle\int_0^r e^{\beta(u)} \delta(u) u^2 du \end{cases}$$

and

$$\begin{cases} \widehat{p}_{\mathrm{rad}}(\delta, \eta) = \delta \partial_\delta \widehat{\rho}(\delta, \eta) - \widehat{\rho}(\delta, \eta) \\ \widehat{p}_{\mathrm{tan}}(\delta, \eta) = \widehat{p}_{\mathrm{rad}}(\delta, \eta) + \dfrac{3}{2} \eta \partial_\eta \widehat{\rho}(\delta, \eta) \end{cases}$$

(so all matter quantities depend on δ).

- Must be careful to consider only physically reasonable solutions when integrating the TOV equations.

- Must check energy conditions:

$$\begin{cases} \text{SEC}: & \rho + p_{\text{rad}} + 2p_{\text{tan}} \geq 0; \quad \rho + p_{\text{rad}} \geq 0; \quad \rho + p_{\text{tan}} \geq 0. \\ \text{WEC}: & \rho \geq 0; \quad \rho + p_{\text{rad}} \geq 0; \quad \rho + p_{\text{tan}} \geq 0. \\ \text{NEC}: & \rho + p_{\text{rad}} \geq 0; \quad \rho + p_{\text{tan}} \geq 0. \\ \text{DEC}: & \rho \geq |p_{\text{rad}}|; \quad \rho \geq |p_{\text{tan}}|. \end{cases}$$

- Must check reality and subluminality of the speeds of sound:

$$\begin{cases} c_{\mathsf{L}}^2(\delta,\eta) = \dfrac{\delta\partial_\delta \widehat{p}_{\mathsf{rad}}}{\widehat{\rho} + \widehat{p}_{\mathsf{rad}}}; \\[2ex] c_{\mathsf{T}}^2(\delta,\eta) = \dfrac{\widehat{p}_{\mathsf{tan}} - \widehat{p}_{\mathsf{rad}}}{(\widehat{\rho} + \widehat{p}_{\mathsf{tan}})\left(1 - \delta^2/\eta^2\right)}; \\[2ex] \tilde{c}_{\mathsf{L}}^2(\delta,\eta) = \dfrac{\delta\partial_\delta \widehat{p}_{\mathsf{tan}} + 3\eta\partial_\eta \widehat{p}_{\mathsf{tan}}}{\widehat{\rho} + \widehat{p}_{\mathsf{tan}}}; \\[2ex] \tilde{c}_{\mathsf{T}}^2(\delta,\eta) = \dfrac{\widehat{p}_{\mathsf{rad}} - \widehat{p}_{\mathsf{tan}}}{(\widehat{\rho} + \widehat{p}_{\mathsf{rad}})\left(1 - \eta^2/\delta^2\right)} \end{cases}$$

(many times missed when considering anisotropic models).

Material models

- Polytropic fluid:

$$\widehat{\rho}(\delta,\eta) = (1-\kappa n)\rho_0\delta + \kappa n\rho_0\delta^{1+\frac{1}{n}}.$$

- Leads to

$$\widehat{p}_{\mathrm{rad}} = \widehat{p}_{\mathrm{tan}} = K\widehat{\sigma}^{1+\frac{1}{n}},$$

with $\widehat{\sigma}$ the baryon density and $K = \kappa(1-\kappa n)^{-\frac{n+1}{n}}\rho_0^{-\frac{1}{n}}$.

- Quadratic elastic correction:

$$\widehat{\rho}(\delta,\eta) = (1-\kappa n)\rho_0\delta + \kappa n\rho_0\delta^{1+\frac{1}{n}} + \varepsilon\rho_0(\delta-\eta)^2.$$

- Model with scale-invariant Newtonian limit:

$$\widehat{\rho}(\delta,\eta) = \rho_0(1-\kappa n)\delta + \kappa\rho_0\delta\eta^{\frac{1}{n}}\Bigg[-\frac{s}{n}\left(\left(1-\frac{n}{s}\right)(1+n)+\frac{2n\varepsilon}{\kappa}\right) + \frac{s}{(1+s)n}\left(1-\frac{n}{s}+\frac{2n\varepsilon}{\kappa}\right)\left(\frac{\delta}{\eta}\right)^{-1} + \frac{s^2}{(1+s)n}\left(1+n+\frac{2n\varepsilon}{\kappa}\right)\left(\frac{\delta}{\eta}\right)^{\frac{1}{s}}\Bigg]$$

- Here s can be interpreted as a shear index; when $s=n$ and $\varepsilon=0$ we recover the relativistic polytropes; for $s=n=1$ it reduces to the quadratic correction model.

- Gauge invariance: these materials are pre-stressed, so no natural reference state; invariance under redefinition of reference state reduces $(\rho_0, \kappa, \varepsilon)$ to two gauge-invariant parameters:

$$K = \kappa(1-\kappa n)^{-\frac{n+1}{n}} \rho_0^{-\frac{1}{n}}, \qquad E = \frac{\varepsilon}{\kappa}\left(\frac{\kappa}{1-\kappa n}\right)^{1-n} \quad \text{or} \quad E = \frac{\varepsilon}{\kappa}.$$

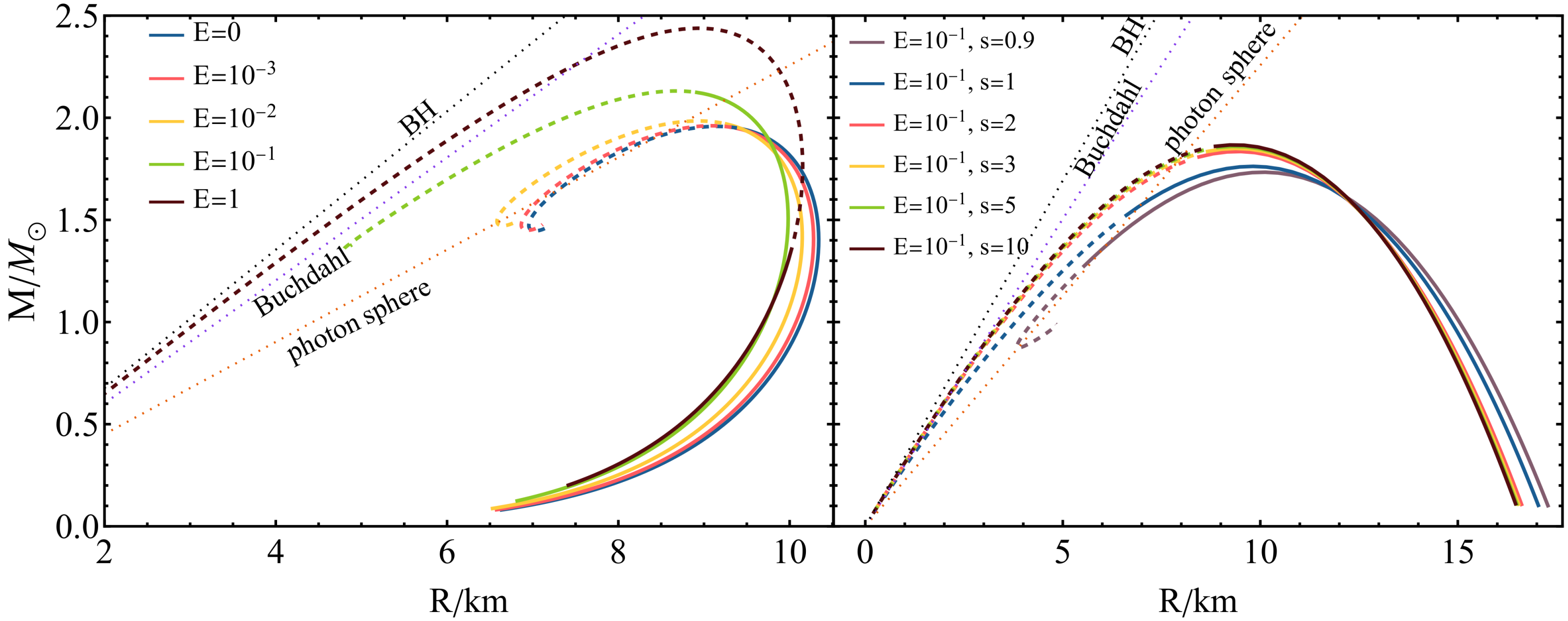
- Radial perturbations: Linearized time-dependent equations for $\zeta(t,r) = e^{i\omega t}\zeta(r)$ and $\chi(t,r) = e^{i\omega t}\chi(r)$ (related to radial displacement and radial pressure) are

$$\begin{cases} \delta\partial_\delta \widehat{p}_{\mathrm{rad}} \dfrac{d\zeta}{dr} & = e^{-(3\alpha+\beta)} r^2 \chi - \frac{3}{r}\eta\partial_\eta \widehat{p}_{\mathrm{rad}}\zeta \\ \delta\partial_\delta \widehat{p}_{\mathrm{rad}} \dfrac{d\chi}{dr} & = \frac{3}{r}\eta\partial_\eta \widehat{p}_{\mathrm{rad}}\chi - \left[Q_1 + Q_2\omega^2\right]\zeta \end{cases},$$

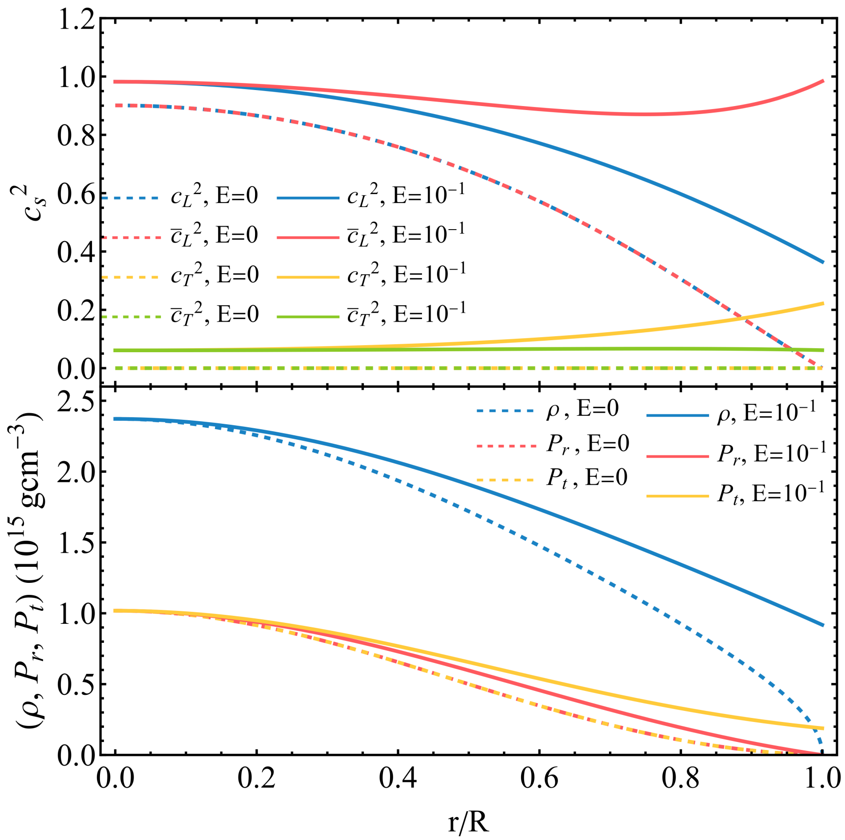
with suitable boundary conditions.

Numerical results

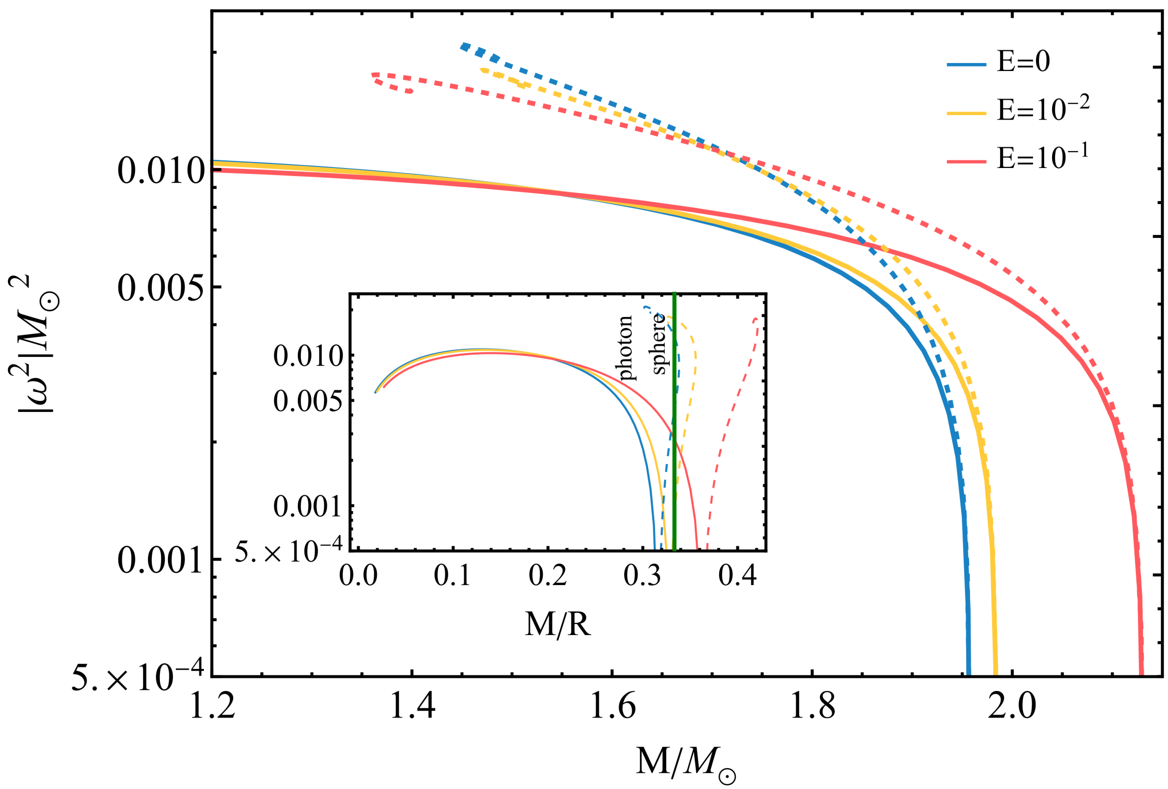
- Mass-radius diagrams for quadratic model with $n = \frac{1}{2}$ (left) and scale-invariant model with $n = 1$ (right). Dashed means superluminal. There exist ultracompact configurations.



- Sound speeds (top) and density/pressure profiles (bottom) for marginally causal stars in the quadratic model with $n = \frac{1}{2}$.



- Radial stability: Squared frequency eigenvalues for radial stability analysis of elastic stars in quadratic model with $n = \frac{1}{2}$. Threshold of instability occurs at the maximum mass.



Conclusion and outlook

- Elastic models are useful tools to model extended bodies in general relativity.

- Anisotropies are often introduced via ad-hoc models, possibly featuring pathologies (energy conditions, speeds of sound). We presented a general yet pratical framework to build physically reasonable anisotropic compact objects within relativistic elasticity.

- We found examples of ultracompact objects (featuring a light ring, $\frac{M}{R} > \frac{1}{3}$), but no violations of Buchdahl's limit $\frac{M}{R} < \frac{4}{9}$. May be interesting to study gravitational wave echoes and nonlinear instabilities due to the second light ring (whose existence we confirmed).

- Still to do:

  - Other models, including non-flat reference metrics;

  - Multilayer solutions;

  - Deformations of generic barotropic fluids;

  - Less symmetric configurations, e.g. rotating and deformed elastic solutions.

Thank you for your attention!